A Text Book of

CO-ORDINATE GEOMETRY

Dr. Ram Ballabh

Prakashan Kendra, Lucknow

Faisal Anjum

A Text Book

of

Co-ordinate Geometry

(For B.A., B.Sc. and Hons. Students)

By

Dr. Ram Ballabh
M. Sc., Ph-D., F.N.A.Sc.
Ex-professor and Head of the Department
of Mathematics and Astronomy
Lucknow University, Lucknow

[13th Revised Edition]

PRAKASHAN KENDRA
Daliganj Railway Crossing, Sitapur Road, Lucknow
☎ (0522) : 2743208, 2743217
e-mail : prakashankendra1952@gmail.com
prakashankendra@rediffmail.com

प्रकाशन केन्द्र लखनऊ

Dedicated to

Shri PADMADHAR MALVIYA

as a token of deep reverence, nobility and system for introducing scientific management as a Director of Text-book Bureau, PRAKASHAN KENDRA, Lucknow

■ Published by :

PRAKASHAN KENDRA

Daliganj Railway Crossing, Sitapur Road, Lko– 226 020.

☎ (0522) : 2743208, 2743217

e-mail : prakashankendra1952@gmail.com
prakashankendra@rediffmail.com

■ Book Code No **393**

■ Price :
Rupees One Hundred Forty & Paise Fifty **(Rs. 140.50)** only.

PUBLISHER'S NOTE TO 13th EDITION

The present edition of this popular book has been thoroughly revised and enlarged to cover the syllabus of various Indian Universities and in the light of suggestions received from readers and well-wishers. New examples have also been added at places where necessary, making the subject-matter more comprehensive. The questions set in recent examination papers of Indian Universities and I. A. S., have also been incorporated to make the book more useful and better suited to meet the ends of students.

-The Publisher.

Scanned by CamScanner

PREEACE TO EIGHTH EDITION

The book has been carefully revised in the light of suggesions received from several friends. A few articles and examples have been added to make the subject-matter more comprehensive and better suited to meet the needs of students.

University of Lucknow —*Ram Ballabh.*

PREFACE TO FOURTH EDITION

The present edition is an improvement upon the third edition as besides correcting the misprints appearing in the earlier edition. New examples have been added at places making the subject-matte more comprehensive. The appendix at the end has been enlarged by the addition of a note on cylindrical and spherical coordinate of solid geometry, so often needed in practical applications of mathematics.

Grateful thanks are due to several friends whose valuable suggestions have been borne in mind while preparing the text of this edition. I shall be failing in may duty if I do not make special mention of the name of Dr. Sahib Ram Mandan of I. I. T Kharagpur, whose suggestions I found extremely useful.

University of Lucknow —*Ram Ballabh*

PREFACE TO THIRD EDITION

The book has been revised and rewritten at places. To make the subject-matter complete for the Honours students, a new chapter on General Conicoids has been added.

University of Lucknow —*Ram Ballabh.*

PREFACE OF SECOND EDITION

Additions and alterations have been made at places and a new chapter on Cone and Cylinder has been added.

University of Lucknow —*Ram Ballabh.*

PREFACE TO FIRST EDITION

This book has been written to meet the requirements of Undergraduates and Honours students of Indian Universities. The treatment of the subject is such that a beginner will find no diffi. culty in following the book. A number of solved examples have been included to illustrate the principles of analytical geometry and to enable the students to attempt successfully the unsolved examples that follow.

A special feature of the book is a separate chapter on the General Equation of the Second Degree, in which loci common to special conics have been obtained. This has the advantage of not only avoiding repetition but also making the student realize the extreme usefulness of the general treatment.

The book contains a large number of examples which have been selected either from the question papers of various Indian and Foreign Universities and of Public Service Examinations or from the standard works on Coordinate geometry.

I take this opportunity to thank the authorities of London Univertity for their kind permission to include the book questions set at various examinations of their University. I also wish to acknowledge my indebtedness to the authors of the existing treatises on the subject which have been freely consulted while writing the present book.

My thanks are also due to several friends who have given their valuable time in reading the manuscript and correcting the proofs.

All suggestions for improvement will be gratefully acknowledged.

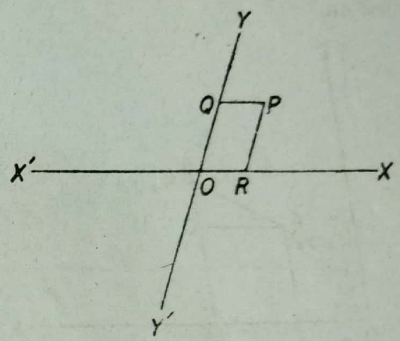
University of Lucknow —*Ram Ballabh.*

CONTENTS

CHAPTER 1

COORDINATES

1.1 Position of a Point. In order to fix the position of a point P in a plane we take two fixed straight lines $X'OX$, $Y'OY$ in the plane and draw PQ, PR parallel to $X'OX$, $Y'OY$ respectively.



Knowing PQ (or OR) and PR (or OQ) we determine P **uniquely.** The lines $X'OX$ and $Y'OY$ are called the **coordinate axes**, $X'OX$ being known as the *axis of x* and $Y'OY$ the *axis of y*. The point O is called the **origin**. The lengths OR $(=PQ)$ and PR $(=OQ)$ are usually denoted by x and y respectively and are called the **abscissa** and the **ordinate** of P. We say that the **coordinates** of P are (x, y), or simply P is the point (x, y).

The angle XOY may have any value between 0 and π. If it is a right angle, the axes are said to be **rectangular.** If it is different from a right angle, the axes are said to be **oblique**.

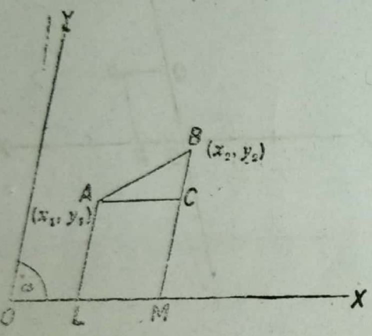
Note. The method of coordinate geometry consists in reducing a geometric problem to an algebraic one which is easier to handle. This method is due to Rene' Descartes after whose name Coordinate Geometry is sometimes called Cartesian Geometry.

As in trigonometry, the plane is supposed to be divided into four quadrants by the axes. The quadrant containing the angle XOY is called the **first quadrant**, the one containing the angle YOX'

the **second quadrant** and the other containing the angles $X'OY'$ and $Y'OX$ the **third** and **fourth** quadrants respectively. *In the first quadrant both x and y are positive, in the second quadrant x is —ve and y +ve, in the third quadrant both x and y are —ve and in the fourth quadrant x is +ve and y —ve.* Thus the points (1, 2), (—3, 4), (—2, —1) and (5, —7) lie respectively in the first, second, third and fourth quadrants.

1.2 Distance between two points.

Let A, B be two given points with coordinates (x_1, y_1), (x_2, y_2) respectively and let the angle between the coordinate axes be ω. *It is required to find AB.*



Draw AL, BM parallel to OY and AC parallel to OX.

Now, $BC=BM-CM=BM-AL=y_2-y_1$.

Similarly, $AC=x_2-x_1$, $\angle ACB=180°-\omega$.

Therefore, from $\triangle ABC$,

$$AB^2=AC^2+BC^2-2AC.BC\cos\angle ACB$$

$$=(x_2-x_1)^2+(y_2-y_1)^2+2\,(x_2-x_1)\,(y_2-y_1)\cos\omega.$$

If $\omega=90°$, *i. e.*, if the axes are **rectangular,**

$$AB^2=(x_2-x_1)^2+(y_2-y_1)^2.$$

The distance AB is thus

$$\pm\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

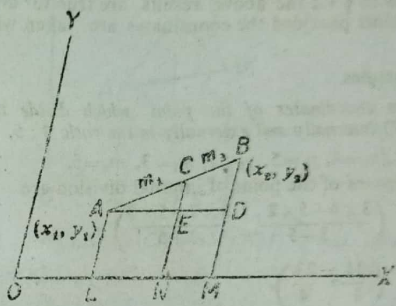
for rectangular axes.

It is customary to express the distance with the **plus sign.** While measuring distances, on the same straight line, the same direction should be considered positive throughout.

Corollary. *The distance of the point (x, y) from the origin is* $\sqrt{x^2+y^2}$ *for rectangular axes.*

Note. The above formula has been proved for the case when both the points lie in the first quadrant. It will however be found to be true for all positions of the points provided the coordinates are taken with proper signs.

1.3 Coordinates of the point dividing the join of two given points in a given ratio.



Let (x_1, y_1), (x_2, y_2) be the coordinates of the given points A and B. Let C divides AB internally in the ratio $m_1 : m_2$ and let $(\bar{x}, \bar{y})$ be the coordinates of C.

Drawing parallels AL, BM, CN to OY meeting AED drawn parallel to OX in A, D and E, we have

$$AE=LN=\bar{x}-x_1,\quad ED=NM=x_2-\bar{x}.$$

From the property of the parallels,

$$\frac{AE}{ED}=\frac{m_1}{m_2},$$

i. e.,
$$\frac{\bar{x}-x_1}{x_2-\bar{x}}=\frac{m_1}{m_2},$$

or
$$\bar{x}=\frac{m_1x_2+m_2x_1}{m_1+m_2}.$$

Similarly,
$$\bar{y}=\frac{m_1y_2+m_2y_1}{m_1+m_2}$$

Corollary. The coordinates of the **point of bisection** of AB are

$$\{\tfrac{1}{2}(x_1+x_2),\ \tfrac{1}{2}(y_1+y_2)\}.$$

External division. If C divides AB externally in the ratio $m_1 : m_2$, the distances AC and CB are directed in opposite senses. If, therefore, m_1 is positive, m_2 is negative and *vice versa.* Using the result for internal division, the coordinates of C for *external division* are easily found to be

$$\left(\frac{m_1x_2-m_2x_1}{m_1-m_2},\ \frac{m_1y_2-m_2y_1}{m_1-m_2}\right).$$

Note 1. The student should deduce that the point of external division lies to the right or left of A according as $m_1 >$ or $< m_2$.

Note 2. As in § 1.2 the above results are true for all positions of the given points provided the coordinates are taken with proper signs.

Solved Examples.

1. Find the coordinates of the point which divide the join of (2, 4) and (5, 7) internally and externally in the ratio 3 : 5.

Here $x_1=2$, $x_2=4$, $y_1=5$, $y_2=7$, $m_1=3$, $m_2=5$.

The coordinates of the point of internal division are

$$\left(\frac{3\times4+5\times2}{3+5},\ \frac{3\times7+5\times5}{3+5}\right),$$

i.e., $$\left(\frac{11}{4},\ \frac{23}{4}\right).$$

and the coordinates of the point of external division are

$$\left(\frac{3\times4-5\times2}{3-5},\ \frac{3\times7-5\times5}{3-5}\right),$$

i.e., $$(-1, 2).$$

2. A circle passes through the points (1, 2), (3, 8) and (17, −6). Find the coordinates of its centre, assuming the axes to be rectangular.

The circum-centre is equidistant from each vertex. If, therefore, (x, y) be the required coordinates, then

$$(x-1)^2+(y-2)^2=(x-3)^2+(y-8)^2.$$
$$=(x-17)^2+(y+6)^2.$$

From these,

$$x+3y=17,$$

and $$2x-y=20.$$

Solving, $x=11$, $y=2$.

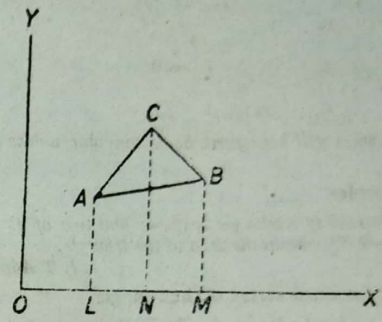
1.4 Area of a triangle.

Let ABC be the triangle and let the coordinates of the vertices A, B, C referred to rectangular axes be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively. *It is required to express the area in terms of the coordinates.*

Draw the ordinates AL, BM, CN.

Now, $\triangle ABC$,

=trapezium AN+trapezium CM−trapezium AM



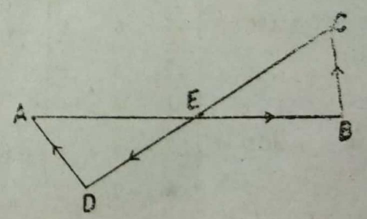
$$=\tfrac{1}{2}(AL+CN).LN+\tfrac{1}{2}(CN+BM).NM-\tfrac{1}{2}(AL+BM).LM$$
$$=\tfrac{1}{2}\{(y_1+y_3)(x_3-x_1)+(y_3+y_2)(x_2-x_3)-(y_1+y_2)(x_2-x_1)\}$$
$$=\tfrac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}.$$

Expressed as a determinant,

$$\triangle ABC=\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note 1. The area is reckoned positive or negative according as it lies to the left or right of an observer going round the triangle from A to C through B. Thus the area of the triangle ABC is positive and of the same triangle written as ABC is negative. *The sign is disregarded whenever numerical value of area is required.*

Note 2. The area of a polygon is determined by breaking it up into triangles. The areas of the triangles thus obtained are added up with due regard to sign. For example, in the figure, the sense of description of the area $A\ B\ C\ D$, sometimes called a quadrilateral, is shown by arrows.



The area is the *difference* (not *sum*) of the areas of the triangles EBC and EDA.

Corollary. Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are **collinear**, if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0.$$

Note. *The axes will henceforth be rectangular unless stated otherwise.*

Solved Examples.

1. If the centroid of a triangle is (1, 4) and two of its vertices are (4, −8) and (−9, 7), obtain the area of the triangle.

[*I. I. T Admission, 1958*]

If (α, β) be the third vertex of the triangle,

$$\frac{\alpha+4-9}{3}=1,\ \frac{\beta-8+7}{3}=4.$$

From these, $\alpha=8$, $\beta=13$.

The area of the triangle is

$$\frac{1}{2}\begin{vmatrix} 8 & 13 & 1 \\ 4 & -8 & 1 \\ -9 & 7 & 1 \end{vmatrix}=166.5 \text{ square units.}$$

2. If the area of the quadrilateral whose angular points A, B, C, D taken in order, are (1, 2), (−5, 6), (7, −4) and (k, −2) be zero, find the value of k.

The diagonal AC divides the quadrilateral into triangles ABC and ACD which are described in the same sense for an observer going round the quadrilateral from A to B, B to C and C to D. Now

$$\Delta ABC=\frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ -5 & 6 & 1 \\ 7 & -4 & 1 \end{vmatrix}=6 \text{ square units,}$$

and $$\Delta ACD=\frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 7 & -4 & 1 \\ k & -2 & 2 \end{vmatrix}=(3k-15) \text{ square units.}$$

From the question,

$$6+3k-15=0,$$

giving $k=3$.

3. The cartesian coordinates (x, y) of a point on a curve are given by

$$x : y : 1=t^3 : t^2-3 : t-1,$$

where t is a parameter. Show that the points given by t=a, b, c are collinear, if

$$abc-(bc+ca+ab)+3\,(a+b+c)=0.$$

[*Math. Tripos, 1946*]

The condition of collinearity is

$$\begin{vmatrix} a^3 & a^2-3 & a-1 \\ b^3 & b^2-3 & b-1 \\ c^3 & c^2-3 & c-1 \end{vmatrix}=0,$$

or $$\begin{vmatrix} a^3-b^3 & a^2-b^2 & a-b \\ b^3-c^3 & b^2-c^2 & b-c \\ c^3 & c^2-3 & c-1 \end{vmatrix}=0,$$

or $$\begin{vmatrix} a^2+ab+b^2 & a+b & 1 \\ b^2+bc+c^2 & b+c & 1 \\ c^3 & c^2-3 & c-1 \end{vmatrix}=0,$$

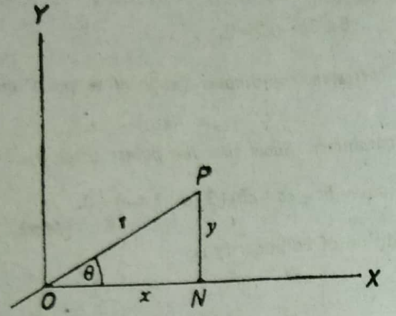
or $$\begin{vmatrix} a^2-c^2+b\,(a-c) & a-c & 0 \\ b^2+bc+c^2 & b+c & 1 \\ c^3 & c^2-3 & c-1 \end{vmatrix}=0,$$

or $$\begin{vmatrix} a+c+b & 1 & 0 \\ b^2+bc+c^2 & b+c & 1 \\ c^3 & c^2-3 & c-1 \end{vmatrix}=0,$$

or $$abc-(bc+ca+ab)+3\,(a+b+c)=0.$$

1.5 Polar coordinates.

Another method often used for fixing the position of a point in a plane is to take a fixed point O called **the pole** and a fixed straight line OX called the **initial line.**



If the angle XOP be θ, and the distance OP be r, the polar coordinates of P are denoted symbolically by (r, θ). r is called the **radius vector** and θ the **vectorial angle** of P.

In expressing the polar coordinates of a given point the *radius vector is always written as the first coordinate.* It is considered positive if measured from the pole along the line bounding the vectorial angle.

In polar coordinates the same point has an infinite number of representations. For example, P has the coordinates

$$(r, \theta), (-r, \theta+\pi), (-r, \theta-\pi), (r, \theta-2\pi)$$

or any set obtained from these by going round the pole any number of times in the appropriate direction. The coordinates of P can, therefore, also be written as

$$(r, \theta+2\pi), (-r, \theta+3\pi), (-r, \theta-3\pi), (r, \theta-4\pi), \text{ etc}.$$

1.6 Relation between polar and rectangular coordinates.

Referring to the figure of § 1.5 let the pole and the origin be the same point O and let the x-axis coincide with the initial line.

The cartesian coordinates of P are then (x, y). If PN be perpendicular from P on OX, then from the right-angled triangle PON,

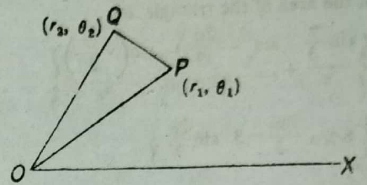
$$x=r\cos\theta, \qquad y=r\sin\theta.$$

From these

$$x^2+y^2=r^2 \text{ and } \tan\theta=\frac{y}{x}.$$

Any relation between x and y can, therefore, be transformed into a relation between r and θ by writing $r\cos\theta$ for x and $r\sin\theta$ for y; and a relation between r and θ can be transformed into a relation between x and y by writing $\sqrt{x^2+y^2}$ for r and $\tan^{-1}\frac{y}{x}$ for θ.

1.7 Distance between two points whose polar coordinates are given.



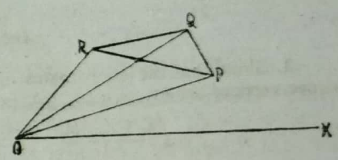
Let P, Q be the given points and let (r_1, θ_1), (r_2, θ_2) be their polar coordinates.

In the triangle OPQ, $\angle POQ=\theta_2-\theta_1$.

and $$PQ^2=OP^2+OQ^2+2OP.OQ\cos\angle POQ$$
$$=r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1).$$

1.8 Area of a triangle in polar coordinates.

Let PQR be the triangle and let the coordinates of the vertices P, Q, R be (r_1, θ_1) (r_2, θ_2), (r_3, θ_3) respectively.



$$\Delta PQR=\Delta OPQ+\Delta OQR-\Delta OPR$$

$$=\tfrac{1}{2}\,OP.OQ\sin \widehat{POQ}+\tfrac{1}{2}\,OQ.OR\sin\widehat{QOR}-\tfrac{1}{2}\,OP.OR\sin\widehat{POR}.$$

$$=\tfrac{1}{2}r_1r_2\sin(\theta_2-\theta_1)+\tfrac{1}{2}r_2r_3\sin(\theta_3-\theta_2)-\tfrac{1}{2}r_1r_3\sin(\theta_3-\theta_1).$$

The area of the triangle is, therefore,

$$\frac{r_1r_2r_3}{2}\left\{\frac{\sin(\theta_3-\theta_2)}{r_1}+\frac{\sin(\theta_1-\theta_3)}{r_2}+\frac{\sin(\theta_2-\theta_1)}{r_3}\right\}$$

Solved Examples.

1. *If x, y be related by means of the equation*

$$(x^2+y^2)^2=a^2(x^2-y^2),$$

find the corresponding relation between r and θ.

Putting $x=r\cos\theta$, $y=r\sin\theta$,
the above relation is transformed into

$$r^4=a^2r^2(\cos^2\theta-\sin^2\theta),$$

i.e.,
$$r^2=a^2\cos 2\theta,$$

2. *Find the area of the triangle whose vertices ar*

$$(c,\theta)\ (2c, \theta+\tfrac{1}{3}\pi), (3c, \theta+\tfrac{2}{3}\pi).$$

From § 1.8, the area of the triangle is

$$\frac{6c^3}{2}\left\{\frac{\sin\frac{\pi}{3}}{c}+\frac{\sin\left(-\frac{2\pi}{3}\right)}{2c}+\frac{\sin\left(\frac{\pi}{3}\right)}{3c}\right\}$$

$$=\frac{c^2}{2}\left\{8\sin\frac{\pi}{3}-3\sin\frac{2\pi}{3}\right\}$$

$$=\frac{5c^2\sqrt{3}}{4}.$$

Examples on Chapter 1

1. State the quadrants in which the following points lie :
$(2, -1)$, $(-4, -5)$, $(3, 7)$, $(-1, 5)$.

2. Find the distance between the following pairs of points :
(*i*) $(2, 3)$, $(5, 7)$; (*ii*) $(3, 2)$, $(-5, -4)$; (*iii*) $(a\cos\alpha, a\sin\alpha)$, $(a\cos\beta, a\sin\beta)$.

Ans. (*i*) 5; (*ii*) 10; (*iii*) $2a\sin\dfrac{\alpha-\beta}{2}$

3. Show that the coordinates of the centroid of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right).$$

Hint. The centroid is the point of intersection of the medians of a triangle.

4. Find the coordinates of the points which divide internally and externally, the line joining $(1, 2)$ to $(4, -5)$ in the ratio $2:3$.

Ans. $\left(\frac{11}{5}, -\frac{4}{5}\right)$, $\left(-5, 16\right)$.

5. Show that the three points $(1, -1)$, $(-\frac{1}{2}, \frac{1}{2})$ and $(1, 2)$ form a right-angled triangle.

Hint. Find the distance between the points taken in pairs.

6. Show that the distance between the points which divide the join of $(4, 3)$ and $(5, 7)$ internally and externally, in the ratio $2:3$ is

$$\frac{12\sqrt{17}}{5} \text{ units.}$$

7. Show that the triangle formed by joining the points

$$\left(\frac{a}{\sqrt{3}}, a\right), \left(\frac{a}{\sqrt{3}}, 3a\right) \text{ and } \left(\frac{4a}{\sqrt{3}}, 2a\right) \text{ is equilateral.}$$

8. Find the coordinates of the circum-centre of the triangle whose vertices are
(*i*) $(2, 1)$ $(2, 3)$ and $(0, 1)$; (*ii*) $(3, 4)$, $(4, 2)$ and $(5, 5)$.

Ans. (*i*) $(1, 2)$; (*ii*) $\left(\frac{9}{2}, \frac{7}{2}\right)$.

9. The coordinates of the vertices A, B, C of a triangle ABC are $(1, 3)$, $(2, 5)$ and $(4, 1)$ respectively, and D is the midpoint of BC. Find the coordinates of the point which divides AD internally in the ratio $3:5$.
Ans. $(\frac{7}{4}, 3)$.

10. Show that the four points $(3, 1)$, $(4, 2)$, $(3, 3)$ and $(2, 2)$ are the angular points of a square.

11. Show that $(2, 1)$, $(4, -1)$, $(-1, -2)$ and $(1, -4)$ are the angular points of a parallelogram.

12. The point (x, y) is equidistant from the points $(-1, 1)$ and $(3, -2)$; show that $8x-6y+11=0$.

13. Find the areas of the triangles formed by joining the following points :
(*i*) $(2, 3)$, $(-4, 7)$, $(8, -3)$; (*ii*) $(1, 1)$, $(2, -1)$, $(-1, 3)$;
(*iii*) $(1, \sqrt{3})$, $(1, 3\sqrt{3})$, $(4, 2\sqrt{3})$.

Ans. (*i*) 6; (*ii*) 1; (*iii*) $3\sqrt{3}$ sq. units.

14. Show that the points $(4, 3)$, $(5, 1)$ and $(1, 9)$ are collinear.

15. If $(1, 1)$, $(7, -3)$, $(12, 2)$ and $(7, 21)$ are the coordinates of the vertices of a quadrilateral, prove that its area is 132 square units.

16. The coordinates of the centroid of a triangle and those of two of its vertices are respectively $(\frac{2}{3}, 2)$ and $(2, 3)$, $(-1, 2)$. Find its area and the radius of the circle circumscribing it.

Ans. $\frac{5}{2}$, $\frac{\sqrt{5}}{\sqrt{2}}$.

17. Transform the following equations as directed :
(*i*) $r^2\cos^2\theta=a^2\operatorname{Cos}2\theta$ to Cartesian coordinates.
(*ii*) $xy^3+yx^3=a^2$ to polar coor...es

Ans. (*i*) $x^2(x^2+y^2)=a^2(x^2-y^2)$,
(*ii*) $r^4=2a^2\operatorname{cosec}2\theta$,

18. In any triangle ABC, prove that

$$AB^2+AC^2=2(AD^2+DC^2),$$

where D is the middle point of BC.

Solution. Take BC and BA as coordinate axes. The coordinates of A are $(0, c)$, and those of C are $(a, 0)$. D is the point $\left(\frac{a}{2}, 0\right)$.

Now, $AC^2=a^2+c^2-2ac\cos B$

and $AD^2=c^2+\left(\frac{a}{2}\right)^2-2.\ \frac{a}{2}.\ c\cos B.$

Therefore, $AB^2+AC^2=a^2+2c^2-2ac\cos B$ ...(1)

and $2\,(AD^2+DC^2)=2c^2+\frac{a^2}{2}-2ac\cos B+2.\frac{a^2}{4}$

$=a^2+2c^2-2ac\cos B$ ...(2)

(1) and (2) being the same, the result is established. It should be remember that *the particular choice of coordinates axes does not make the solution less general*.

19. G is the centroid of a triangle ABC and O any other point, prove that

(*i*) $3\,(GA^2+GB^2+GC^2)=BC^2+CA^2+AB^2$,

(*ii*) $OA^2+OB^2+OC^2=GA^2+GB^2+GC^2+3GO^2$.

Hint. Take AB and AC as coordinate axes.

20. Show that the coordinates of the point of intersection of any two of the internal bisectors of the angles of a triangle whose vertices are (α_1, β_1), (α_2, β_2) and (α_3, β_3) are

$$\frac{a\alpha_1+b\alpha_2+c\alpha_3}{a+b+c} \text{ and } \frac{a\beta_1+b\beta_2+c\beta_3}{a+b+c},$$

where a, b, c are lengths of sides of the triangle. Hence establish the concurrence of the internal bisectors.

Hint. The internal bisectors of an angle of a triangle divides the opposite side in the ratio of the sides including the angle.

21. The coordinates of the middle points of the sides of a triangle being given, find the vertices.

Hint. If (α_1, β_1), (α_2, β_2), (α_3, β_3) be the coordinates of the middle points, and (x_1, y_1), (x_2, y_2), (x_3, y_3) those of the vertices, $x_2+x_3=2\alpha_1$, etc.

CHAPTER II

LOCUS

2.1 Definition. *The* **locus** *of a point is the path traced by the point when it moves in accordance with certain given conditions.* Thus, for example, if a point moves such that it is at equal distances from two fixed points in its plane of motion, it will lie on the straight line bisecting the join of the fixed points at right angles. We say that the locus of the point is this right bisector. If, on the other hand, the point moves in a plane such that its distance from a fixed point in the plane always remains the same, the locus of the point will be a circle with point as its centre.

2.2 Equation to a curve. *The relation which exists between the coordinates of any point on a curve is called the equation to the curve.* For example, if (x, y) be the coordinates of any point on the straight line joining the points (x_1, y_1), (x_2, y_2), its equation will be

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0,$$

for, this is the relation between x and y obtained by considering the area of the triangle formed by these points.

The equation to a curve expresses the law governing all points which lie on that curve, and points not obeying that law, will not satisfy the equation to the curve.

A curve is a locus and its equation is therefore the equation to the locus. The converse is, however, not true. For example, the locus of points of which the y coordinate is positive, is the upper half xy-plane which cannot be expressed in the form of an equation. The same is true of points of which the distances from a fixed point are less than a given distance. In this latter case, the points lie within the circumference of a fixed circle and cannot be represented collectively by an equation.

2.3 Equation of a curve and coordinate axes. The equation of a curve depends upon the choice of coordinate axes. For example, the equations

$$(x-h)^2+(y-k)^2=a^2,$$

and $$x^2+y^2=a^2$$

represent a circle of radius a. The two circles are identical except for the coordinates of their centres. By a suitable choice of the

Solution. Take BC and BA as coordinate axes. The coordinates of A are $(0, c)$, and those of C are $(a, 0)$. D is the point $\left(\frac{a}{2}, 0\right)$.

Now, $AC^2=a^2+c^2-2ac \cos B$

and $AD^2=c^2+\left(\frac{a}{2}\right)^2-2.\ \frac{a}{2}.\ c \cos B.$

Therefore, $AB^2+AC^2=a^2+2c^2-2ac \cos B$ ...(1)

and $2\,(AD^2+DC^2)=2c^2+\frac{a^2}{2}-2ac \cos B+2.\frac{a^2}{4}$

$=a^2+2c^2-2ac \cos B$ ...(2)

(1) and (2) being the same, the result is established. It should be remember that *the particular choice of coordinates axes does not make the solution less general*.

19. G is the centroid of a triangle ABC and O any other point, prove that

(*i*) $3\,(GA^2+GB^2+GC^2)=BC^2+CA^2+AB^2$,

(*ii*) $OA^2+OB^2+OC^2=GA^2+GB^2+GC^2+3GO^2$.

Hint. Take AB and AC as coordinate axes.

20. Show that the coordinates of the point of intersection of any two of the internal bisectors of the angles of a triangle whose vertices are (α_1, β_1), (α_2, β_2) and (α_3, β_3) are

$$\frac{a\alpha_1+b\alpha_2+c\alpha_3}{a+b+c} \text{ and } \frac{a\beta_1+b\beta_2+c\beta_3}{a+b+c},$$

where a, b, c are lengths of sides of the triangle. Hence establish the concurrence of the internal bisectors.

Hint. The internal bisectors of an angle of a triangle divides the opposite side in the ratio of the sides including the angle.

21. The coordinates of the middle points of the sides of a triangle being given, find the vertices.

Hint. If (α_1, β_1), (α_2, β_2), (α_3, β_3) be the coordinates of the middle points, and (x_1, y_1), (x_2, y_2), (x_3, y_3) those of the vertices, $x_2+x_3=2\alpha_1$, etc.

CHAPTER II

LOCUS

2.1 Definition. *The* **locus** *of a point is the path traced by the point when it moves in accordance with certain given conditions.* Thus, for example, if a point moves such that it is at equal distances from two fixed points in its plane of motion, it will lie on the straight line bisecting the join of the fixed points at right angles. We say that the locus of the point is this right bisector. If, on the other hand, the point moves in a plane such that its distance from a fixed point in the plane always remains the same, the locus of the point will be a circle with point as its centre.

2.2 Equation to a curve. *The relation which exists between the coordinates of any point on a curve is called the equation to the curve.* For example, if (x, y) be the coordinates of any point on the straight line joining the points (x_1, y_1), (x_2, y_2), its equation will be

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0,$$

for, this is the relation between x and y obtained by considering the area of the triangle formed by these points.

The equation to a curve expresses the law governing all points which lie on that curve, and points not obeying that law, will not satisfy the equation to the curve.

A curve is a locus and its equation is therefore the equation to the locus. The converse is, however, not true. For example, the locus of points of which the y coordinate is positive, is the upper half xy-plane which cannot be expressed in the form of an equation. The same is true of points of which the distances from a fixed point are less than a given distance. In this latter case, the points lie within the circumference of a fixed circle and cannot be represented collectively by an equation.

2.3 Equation of a curve and coordinate axes. The equation of a curve depends upon the choice of coordinate axes. For example, the equations

$$(x-h)^2+(y-k)^2=a^2,$$

and

$$x^2+y^2=a^2$$

represent a circle of radius a. The two circles are identical except for the coordinates of their centres. By a suitable choice of the

coordinate axes, we can have the second simpler equation of the circle instead of the first.

In working out problems, the coordinate axes, if not specified, should always be chosen such that unnecessary calculation is avoided and the equation of the curve is obtained in as simple a form as possible.

Note. To obtain the equation to a curve or locus, it is customary to assume the coordinates of any point on the locus as different from x, y which are treated as current coordinates. The coordinates of the point are changed to current coordinates after the given law has been expressed algebraically. After working out a few examples, the student may conveniently ignore this practice and assume x, y as the coordinates of any point on the locus.

We shall now give a few simple examples on the formation of the equation to a locus.

Solved Examples.

1. Find the equation to the locus of a point which moves such that its distance from the point (4, 2) is three times its distance from the point (2, 1).

Let (x, y) be any position of the moving point. Then, by the condition of the question,

$$(x-4)^2+(y-2)^2=9\{(x-2)^2+(y-1)^2\}$$

i.e., $$8x^2+8y^2-28x-14y+25=0.$$

This being the relation between the coordinates of any point that satisfies the given condition is the equation to the required locus.

2. A and B are two fixed points (a, 0) and (−a, 0) respectively; obtain the equation to the locus of P when

$$PA^2-PB^2=4a^2.$$

Let (x, y) be any point which satisfies the given condition. We then have

$$(x-a)^2+y^2-\{(x+a)^2+y^2\}=4a^2.$$

i.e., $$-4ax=4a^2,$$

or $$x+a=0.$$

This being the condition satisfied by P is the equation to the locus of P.

Examples on Chapter II

1. Prove that the locus of a point which is equidistant from the points $(a+b, b-a)$ and $(a-b, a+b)$ is $bx-ay=0$.

2. Find the equation to the locus of a point whose distance from the point $(a, 0)$ is equal to its distance from the y-axis.
Ans. $y^2-2ax+a^2=0$.

3. Find the equation to the locus of a point whose distance from the origin exceeds its distance from the axis of x by 2.
Ans. $x^2=4(y+1)$.

4. P and Q are two variable points on the axes of x and y respectively, such that $OP+OQ=a$; find the equation of the locus of the foot of the perpendicular from the origin on PQ.
Ans. $(x+y)(x^2+y^2)=axy$.

5. A stick of length l rests against the floor, and a wall of a room. If the stick begins to slide on the floor, find the locus of its middle point.
Ans. $4(x^2+y^2)=l^2$.

6. A point moves such that the difference of its distances from the points $(c, 0)$, $(-c, 0)$ is $2a$. Determine its locus.
Ans. $\dfrac{x^2}{a^2}-\dfrac{y^2}{c^2-a^2}=1$.

7. A straight line AB, of fixed length, slides between two perpendicular lines OX and OY, in such a way that the point A always lies on OX and the point B on OY. Find the locus of the point P which divides AB into two parts PA and PB such that $PA=a$ and $PB=b$.
Ans. $a^2x^2+b^2y^2=a^2b^2$.

8. Two points A and B have the coordinates $(1, 0)$ and $(-1, 0)$ respectively. Find equations for the loci of points P and Q which are such that

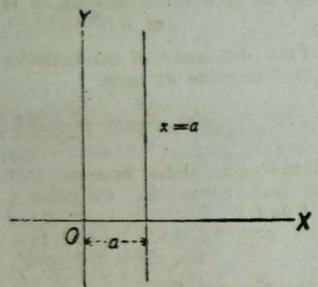
$$AP+BP=4;$$
$$AQ-BQ=\pm 1.$$

Ans. $3x^2+4y^2=12$; $12x^2-4y^2=3$.

CHAPTER III

THE STRAIGHT LINE

3.1 Equation of a straight line. We shell find the equation of a straight line by considering any variable point on it and establishing the relation between the coordinates of this point. As simple cases we may take straight lines drawn parallel to coordinate axes. A straight line drawn parallel to the axis of y at a distance a from it, will be such that the abscissa of any point on the line will have the value a.

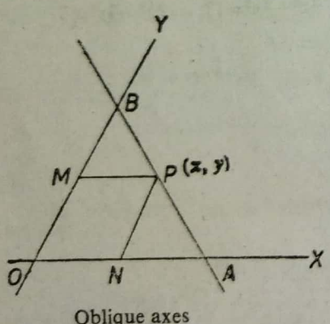


The equation of the line will therefore be $\mathbf{x=a}$. *Similarly, the equation of a straight line drawn parallel to the x-axis at a distance* **b** *from it will be* $\mathbf{y=b}$.

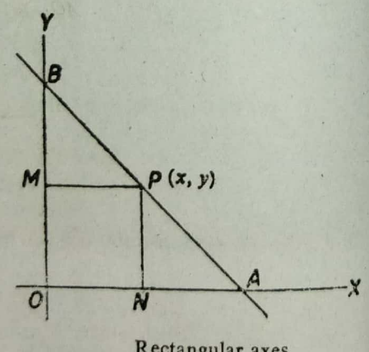
Corollary. The equations to the axes of x and y are respectively $\mathbf{y=0}$ and $\mathbf{x=0}$.

Note. The student should verify that for oblique axes the equations to straight lines drawn parallel to coordinate axes are *exactly* of the same form as for rectangular axes.

3.1.1. Equation to straight line when the intercepts on the axes are given.



Oblique axes



Rectangular axes

Let the given straight line cut off intercepts OA and OB from the axes of x and y respectively. Let $OA=a$ and $OB=b$.

Let P be a variable point (x, y) on the line. Draw PN and PM parallel to OY and OX respectively. Evidently triangles PNA and PMB are similar.

Therefore, $$\frac{NA}{MP}=\frac{NP}{MB},$$

or $$\frac{a-x}{x}=\frac{y}{b-y},$$

or $$bx+ay=ab.$$

Dividing by ab, *the equation to the straight line cutting off intercepts a and b from the axes is*

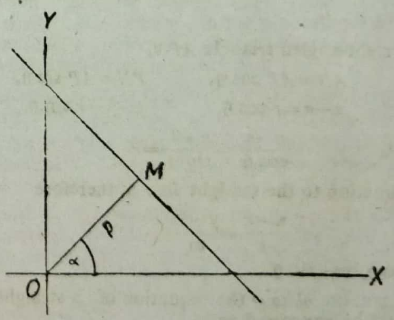
$$\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}=\mathbf{1}.$$

Note. We have found the equation to the straight line by taking the intercepts on the positive sides of the axes. The result is however true for all positions of the line provided we take the intercepts with due regard to sign. For example, the equation

$$\frac{x}{2}-\frac{y}{3}=1$$

represents a straight line which cuts off intercepts 2 from x-axis and -3 from y-axis. This means that the straight line cuts the x-axis on the positive side and the y-axis on the negative side, the distances of the points of intersection from the origin being 2 and 3 units respectively.

3.12 Equation to the straight line when the perpendicular from the origin is given.



Let the length of the perpendicular OM from the origin on the given straight line be p and let OM make an angle α with x-axis.

The intercepts on the coordinate axes are evidently $p \sec \alpha$ and $p \operatorname{cosec} \alpha$. Using the result of the preceding article, the equation of the straight line is seen to be

$$x \cos \alpha + y \sin \alpha = p.$$

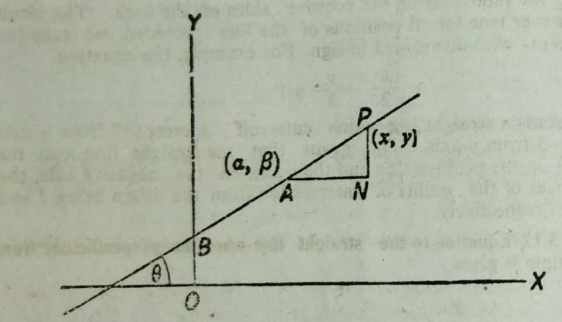
The reader should independently deduce the above result by finding the locus of a moving point on the straight line as an exercise.

Note. In the above case the axes are rectangular.

3.13 Equation when the straight line passes through fixed point and is inclined at a fixed angle to the x-axis.

Let A be the fixed point (α, β) on the given straight line and P a variable point (x, y).

Let θ be the angle which the line makes with the axis of x. Draw AN, PN parallel to the coordinate axes and let $AP=r$.



In the right-angled triangle APN,

$$AN = AP \cos \theta, \qquad PN = AP \sin \theta,$$

i.e., $$x-\alpha = r \cos \theta, \qquad y-\beta = r \sin \theta,$$

or $$\frac{x-\alpha}{\cos \theta} = \frac{y-\beta}{\sin \theta} = r.$$

The equation to the straight line is therefore

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m},$$

where $l=\cos \theta$, $m=\sin \theta$.

If the axes are *oblique* the equation of a straight line through (α, β) can still be expressed as

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = r,$$

where l and m are no longer the cosine and the sine of the angle which the line makes with the x-axis, but constants depending only on the direction of the line.

Note. Distances measured on opposite sides of A have opposite signs.

Corollary. The coordinates of a variable point on the line are given by

$$x = \alpha + lr, \quad y = \beta + mr,$$

3.14 Equation when the inclination to x-axis and intercept on y-axis are given.

Let the straight line be inclined at an angle $\tan^{-1} m$ to the x-axis and cut off an intercept c from the y-axis.

Referring to the figure of the preceding article, $\tan \theta = m$ and $OB = c$.

The coordinates of B are $(0, c)$, which is a fixed point on the straight line. Using the previous result, the equation to the line is

$$\frac{x-0}{\cos \theta} = \frac{y-c}{\sin \theta},$$

or $$y = x \tan \theta + c,$$

i.e., $$y = mx + c.$$

m is called the **slope or gradient** of the line.

Note. The reader should deduce the above result independently by finding the equation to the locus of a moving point on the given straight line.

Corollary. The equation of a straight line passing through the origin is $y = mx$.

3.2 Equation of the first degree. The various forms of the equation of a straight line obtained in the preceding sections are all linear equations, *i.e.*, equations of the first degree in x and y. We shall now prove the following general result :

The equation of straight line is of the first degree and, conversly, that every equation of the first degree represents a straight line.

In proving the above result we shall use the property that a straight line is uniquely determined if the coordinates of two distinct points lying on it, are known.

Let (x_1, y_1), (x_2, y_2) be two fixed points on a straight line and let (x, y) be a variable point. These three points form a triangle of zero area. The relation between x and y, thus, is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

i.e., $x(y_1-y_2)+y(x_2-x_1)+x_1y_2-y_1x_2=0,$

or $Ax+By+C=0,$

where $A=y_1-y_2$, $B=x_2-x_1$, $C=x_1y_2-y_1x_2$.

The above is also the equation to the straight line under consideration (*cf.* § 2.2). This being an equation of the first degree, hence we conclude that *the equation of a straight line is of the first degree.*

Conversely, let $Ax+By+C=0$ be the general equation of the first degree. This is the locus of certain points. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be three distinct points on the locus. These coordinates then satisfy the equation to the locus.

This gives

$$Ax_1+By_1+C=0,$$
$$Ax_2+By_2+C=0,$$

and
$$Ax_3+By_3+C=0.$$

Eliminating A, B, C, we obtain

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0,$$

which expresses the fact that the area of the triangle formed by the three points is zero, *i.e.*, the three points are collinear. The given locus is thus a straight line.

Every equation of the first degree, therefore, represents nothing but a straight line.

Note 1. The reader should verify that the *condition of collinearity* of three points remains the same whether the axes are rectangular or oblique. A first degree equation in x and y, therefore, always represents a straight line, the convers proposition being also true.

Note 2. The equation $Ax+By+C=0$ involves only two arbitrary constants as can easily be seen on dividing by C and treating A/C and B/C as new constants. To evaluate these two constants two conditions satisfied by the line must be given. We say that, written in the general form, the equation of the straight lines has *two degrees of freedom.*

3.21 Equation to the straight line joining two given points.

Let the coordinates of the given points be (x_1, y_1), (x_2, y_2). As in the preceding article, equation to the straight line is

$$x(y_1-y_2)+y(x_2-x_1)+x_1y_2-y_1x_2=0.$$

This can be written as

$$x(y_1-y_2)-x_1(y_1-y_2)=y(x_1-x_2)-y_1(x_1-x_2),$$

or $$y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1).$$

The *slope* of the line is evidently

$$\frac{y_1-y_2}{x_1-x_2}.$$

Corollary. The equation to a straight line passing through the fixed point (x_1, y_1) is $y-y_1=m(x-x_1)$, where m is arbitrary.

Note. The reader should verify that the equation to the straight line joining (x_1, y_1), (x_2, y_2) remains the same for oblique axes.

3.3 Line at infinity. We know that the equation $Ax+By+C=0$ represents a straight line. Writing this as

$$\frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1,$$

we see that the intercepts on the axes are $-\frac{C}{A}$ and $-\frac{C}{B}$. If either $A\to 0$ or $B\to 0$, C remaining finite, the length of the intercept on one coordinate axis becomes infinite and the straight line becomes parallel to that coordinate axis. If, however, both A and B tend to zero simultaneously, the lengths of the intercepts on both the axes tend to infinity.

The equation which then reduces to $C=0$ represents a straight line lying wholly at infinity.

3.4 Line through the intersection of two given lines.

Let $a_1x+b_1y+c_1=0,$ ...(1)

and $a_2x+b_2y+c_2=0,$ ...(2)

be two given lines, and let

$$a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0 \qquad ...(3)$$

where λ is an arbitrary constant, be the equation to any other line.

The values of x and y which satisfy (1) and (2) simultaneously obviously satisfy (3) also, which therefore is a straight line passing through the intersection of (1) and (2). This line will have one degree of freedom, that is, it can be made to satisfy only one arbitrary condition.

Aliter. Let (x', y') be the point of intersection of (1) and (2). Then, since (x', y') lies on both these lines,

$$a_1x'+b_1y'+c_1=0,$$

and
$$a_2x'+b_2y'+c_2=0.$$

Solving these as simultaneous equations,

$$x'=\frac{b_1c_2-c_1b_2}{a_1b_2-a_2b_1}, \quad y'=\frac{c_1a_2-a_1c_2}{a_1b_2-a_2b_1}.$$

The required straight line is $y-y'=m(x-x')$, where m is arbitrary.

If $a_1b_2-a_2b_1 \to 0$, both x' and $y' \to \infty$, provided the lines remain distinct. In this case the lines become parallel, as we shall see in § 3.5.

Note. The student will find that form (3) is more useful in working out the problems.

3.41 Straight line passing through a fixed point.

If the equation to a straight line can be written in the form

$$a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0,$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are fixed and λ variable, the line passes through the point of intersection of the lines

$$a_1x+b_1y+c_1=0,$$

and

$$a_2x+b_2y+c_2=0,$$

which is a fixed point.

An equation of the first degree containing one arbitrary parameter represents a straight line passing through a fixed point.

Examples

1. Find the equation to the straight line joining the origin to the point of intersection of the lines

$$\frac{x}{a}+\frac{y}{b}=1 \text{ and } \frac{x}{b}+\frac{x}{b}=1.$$

Solution.

Any line through the point of intersection of the given lines is

$$\frac{x}{a}+\frac{y}{b}-1+\lambda\left(\frac{x}{b}+\frac{y}{a}-1\right)=0.$$

Since (0, 0) satisfies the equation, $-1-\lambda=0$, or $\lambda=-1$.

The equation to the line is, therefore,

$$\frac{x}{a}+\frac{y}{b}-1-\left(\frac{x}{b}+\frac{y}{a}-1\right)=0.$$

or

$$x=y.$$

2. Obtain the equation of the lines passing through the intersection of $4x-3y-1=0$ and $2x-5y+3=0$ and equally inclined to the axes.

Ans. $x+y=2$; $x=y$.

3. Find the equation to the straight line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$. If t_1, t_2 are roots of the equation

$$t^2+\lambda t+1=0,$$

where λ is arbitrary; show that the line passes through a fixed point.

Solution.

The equation to the line joining the given points is

$$y-2at_1=\frac{2at_1-2at_2}{at_1^2-at_2^2}(x-at_1^2),$$

$$=\frac{2}{t_1+t_2}(x-at_1^2).$$

Simplifying,

$$(t_1+t_2)\,y-2\,(x+at_1t_2)=0.$$

Here we have

$$t_1+t_2=-\lambda, \quad t_1t_2=1.$$

Substituting in the above, we obtain

$$\lambda y+2\,(x+a)=0.$$

This always passes through the point of intersection of $y=0$ and $x+a=0$, which is the fixed point $(-a, 0)$.

4. If a straight line passes through a fixed point, find the locus of the middle point of the portion of it which is intercepted between two given straight lines.

Ans. $\frac{h}{x}+\frac{k}{y}=2.$

Hint. Take (h, k) as the given point and two given lines as coordinate axes.

5. Find the equation of the straight line which passes through the point $(-1, 2)$ and makes equal intercepts on the axes.

Ans. $x+y=1$.

6. Show that the equations of the straight line passing through the point $(1, -1)$ and making angles of $150°$ and $30°$ respectively with the axis of x are

$$y+1=\mp\frac{1}{\sqrt{3}}(x-1).$$

7. A straight line is drawn through the point P (1, 2) and intersects the line $x-2y+4=0$ in Q. If the distance PQ be $\sqrt{2}$, find the direction of the first line.

Solution.

From the corollary to § 3.13, the coordinates (x, y) of a variable point on the first line are given by

$$x=1+lr,\ y=2+mr.$$

If this lies on the second line also,

$$1+lr-2\,(2+mr)+4=0,$$

or

$$r\,(l-2m)+1=0.$$

Here $r=\sqrt{2}$. Therefore,

$$\sqrt{2}\,(l-2m)+1=0,$$

$$\sqrt{2}\cos\theta-2\sqrt{2}\sin\theta+1=0,$$

since l and m are $\cos\theta$ and $\sin\theta$ respectively.

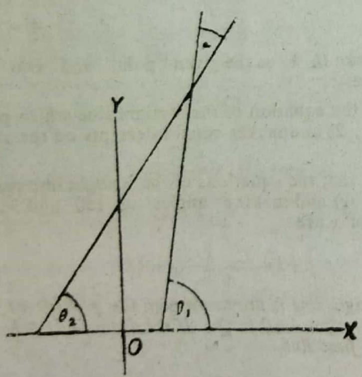
Solving, $\theta=45°$.

8. Given the vertical angle of a triangle in magnitude and position, and the sum of the reciprocals of the sides, show that the base will pass through a fixed point.

Hint. Take the sides of the triangle as coordinate axes.

3.5 Angle between two lines.

Let the straight lines $y=m_1x+c_1$ and $y=m_2x+c_2$ intersect at an angle α. If θ_1 and θ_2 be the inclinations of the lines to x-axis, $\tan\theta_1=m_1$ and $\tan\theta_2=m_2$.



From the figure,

$$\alpha=\theta_1-\theta_2.$$

Hence,

$$\tan\alpha=\tan(\theta_1-\theta_2)$$

$$=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}$$

$$=\frac{m_1-m_2}{1+m_1m_2}.$$

The angle between the lines is, therefore,

$$\tan^{-1}\frac{m_1-m_2}{1+m_1m_2}.$$

Note. In numerical examples the value of $\tan\alpha$ will sometimes be found to be negative. This will merely mean that, instead of getting the acute angle of intersection, its supplement, which, too, is the angle of intersection of the lines, is being obtained. *The result may always be taken with the plus sign ignoring the minus sign whenever it occurs.*

Corollary 1. *The lines $y=m_1x+c_1$ and $y=m_2x+c_2$ are parallel if $m_1=m_2$, and perpendicular if $m_1m_2=-1$.*

Corollary 2. *The lines $a_1x+by_1+c_1=0$ and $a_2x+b_2y+c_2=0$ are parallel if $a_1b_2-a_2b_1=0$ and perpendicular if $a_1a_2+b_1b_2=0$.*

For the slopes of the lines are $-\frac{a_1}{b_1}$ and $-\frac{a_2}{b_2}$.

Corollary 3. The angle between the lines $x\cos\alpha_1+y\sin\alpha_1-p_1=0$ and $x\cos\alpha_2+y\sin\alpha_2-p_2=0$ is $a_1\sim a_2$. The lines are parallel if $\alpha_1=\alpha_2$ and perpendicular, if $\alpha_1\sim\alpha_2=\frac{\pi}{2}$.

3.51 Straight line drawn parallel or perpendicular to a given straight line through a given point.

Let the given straight line be

$$ax+by+c=0 \qquad ...(1)$$

and let the given point be (x_1, y_1).

The slope of (1) is $-\frac{a}{b}$.

Now, any line through (x_1, y_1) is

$$y-y_1=m\,(x-x_1). \qquad ...(2)$$

This is parallel to (1) if $m=\frac{-a}{b}$.

Hence the required *parallell* line is

$$y-y_1=-\frac{a}{b}(x-x_1)$$

or $$a(x-x_1)+b(y-y_1)=0.$$

Again, (2) is perpendicular to (1), if

$$m\left(-\frac{a}{b}\right)=-1,$$

or $$m=\frac{b}{a}.$$

The required *perpendicular line* is, therefore,

$$y-y_1=\frac{b}{a}(x-x_1)$$

or $$a(y-y_1)-b(x-x_1)=0.$$

The above results can be summarised in the form of the following rule:

Replace x by $x-x_1$, y by $y-y_1$ and omit the constant term in the equation of a straight line and you get a parallel straight line through (x_1, y_1).

Replace x by $x-x_1$, y by $y-y_1$, interchange the coefficients of x and y changing the sign of one of the coefficients, and omit the constant term in the equation of a straight line, and you get a perpendicular straight line through (x_1, y_1).

Examples

1. Find the straight line parallel to $2x+3y+1=0$ through the point $(-1, 2)$. *Ans.* $2x+3y=4$.

2. Find the coordinates of the foot of the perpendicular from the point (2, 3) on the line $x+y-11=0$. *Ans.* (5, 6).

3. Find the equation of a straight line which passes through the point of intersection of two given lines and is perpendicular to a third line in their plane.

Prove that the point $(-1, 4)$ is the orthocentre of the triangle which is formed by the lines whose equations are

$$x-y+1=0,\ x-2y+4=0,\ 9x-3y+1=0.$$

4. *Show that the equations of the lines which pass through (a, b) and make an angle θ with the line $px+qy+r=0$ are*

$$\begin{vmatrix} x & y & 1 \\ a & b & 1 \\ p\sin\theta\pm q\cos\theta & p\sin\theta\mp q\cos\theta & 0 \end{vmatrix}=0.$$

Solution.

Let one of the required lines be

$$Ax+By+C=0. \quad ...(1)$$

Since it passes through (a, b), we have

$$Aa+Bb+C=0. \quad ...(2)$$

The slopes of the given line and (1) are

$$-\frac{p}{q} \text{ and } -\frac{A}{B} \text{ respectively.}$$

Therefore

$$\tan\theta=\pm\left\{\left(\frac{p}{q}-\frac{A}{B}\right) \Big/ \left(1+\frac{Ap}{Bq}\right)\right\},$$

i.e., $A(p\sin\theta\pm q\cos\theta)+B(q\sin\theta\mp p\cos\theta)=0. \quad ...(3)$

Eliminating A, B, C from (1), (2) and (3), we get the desired result.

5. The sides AB, BC, CD, DA of a quadrilateral have equations $x+2y=3$, $x=1$, $x-3y=4$, $5x+y+12=0$ respectively. Show that the diagonals AC and BD are at right angles.

Hint. The coordinates of B and C are respectively (1, 1) and $(1, -1)$. Any line through A is $5x+y+12+\lambda(x+2y-3)=0$. This passes through C. Determine λ and proceed.

6. Find the equations to the diagonals of the parallelogram formed by the lines

$$lx+my+n=0, \quad l'x+m'y+n=0,$$
$$lx+my+n'=0, \quad l'x+m'y+n'=0,$$

and show that they are at right angles if $l^2+m^2=l'^2+m'^2$.

Hint. Choose λ and μ such that

$$lx+my+n+\lambda(l'x+m'y+n)=0$$

and $$lx+my+n'+\mu(l'x+m'y+n')=0$$

become the same equation. This gives one diagonal as

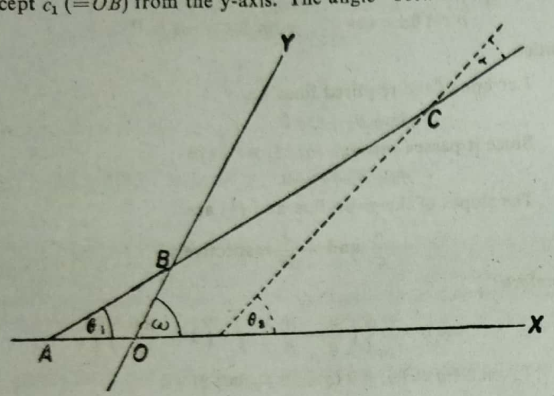
$$(l-l')x+(m-m')y=0.$$

The other diagonal is

$$(l+l')x+(m+m')y+n+n'=0.$$

3.52 Angle between two lines when the axes are oblique.

We shall first find the equation to the straight line AC which is inclined at an angle θ_1 to the axis of x and which cuts off an intercept c_1 $(=OB)$ from the y-axis. The angle between the axes is ω.



In the triangle AOB, $\angle ABO=\omega-\theta_1$.

Therefore,

$$\frac{AO}{\sin(\omega-\theta_1)}=\frac{OB}{\sin\theta_1}=\frac{c_1}{\sin\theta_1},$$

or
$$AO=\frac{c_1\sin(\omega-\theta_1)}{\sin\theta_1}.$$

From § 3.11, the equation to AC is

$$\frac{x}{-AO}+\frac{y}{OB}=1,$$

i.e.,
$$y=m_1x+c_1,$$

where
$$m_1=\frac{OB}{AO}=\frac{\sin\theta_1}{\sin(\omega-\theta_1)}.$$

This gives
$$\tan\theta_1=\frac{m_1\sin\omega}{1+m_1\cos\omega}.$$

Similarly, if $y=m_2x+c_2$ be another line inclined at an angle θ_2 to x-axis,

$$\tan\theta_2=\frac{m_2\sin\omega}{1+m_2\cos\omega}.$$

The angle α between two lines is given by

$$\tan\alpha=\tan(\theta_2-\theta_1)$$
$$=\frac{\tan\theta_2-\tan\theta_1}{1+\tan\theta_2\tan\theta_1}.$$

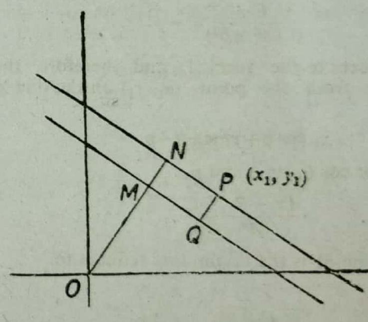
Substituting for $\tan\theta_1$, $\tan\theta_2$ and simplifying, we obtain

$$\alpha=\tan^{-1}\frac{(m_2-m_1)\sin\omega}{1+(m_1+m_2)\cos\omega+m_1m_2}.$$

Corollary. *The lines* $y=m_1x+c_1$ *and* $y=m_2x+c_2$ *are parallel if* $m_1=m_2$ *and perpedicular if* $1+(m_1+m_2)\cos\omega+m_1m_2=0$.

Ex. If the straight lines $y=m_1x+c_1$ and $y=m_2x+c_2$ make equal angles with the axis of x and be not parallel to one another, prove that $m_1+m_2+2m_1m_2\cos\omega=0$, where ω is the angle between the axes.

3.6 Length of the perpendicular from a fixed point on a given straight line.



Let the given line be

$$x\cos\alpha+y\sin\alpha-p=0 \qquad ...(1)$$

and let the fixed point be P with coordinates (x_1, y_1).

The perpendicular OM from the origin on the given line is p.

The equation to a line parallel to (1) through P is

$$(x-x_1)\cos\alpha+(y-y_1)\sin\alpha=0,$$

or
$$x\cos\alpha+y\sin\alpha-(x_1\cos\alpha+y_1\sin\alpha)=0.$$

The perpendicular ON from the origin on this line is, therefore,

$$x_1\cos\alpha+y_1\sin\alpha.$$

If PQ is the perpendicular from P on (1), then from the rectangle $PQMN$,

$$PQ=MN.$$

But
$$MN=ON-OM$$
$$=x_1\cos\alpha+y_1\sin\alpha-p.$$

Hence the length of the perpendicular from (x_1, y_1) *on the line* $x\cos\alpha+y\sin\alpha-p=0$ is

$$x_1\cos\alpha+y\sin\alpha-p.$$

If the equation to the line is

$$Ax+By+C=0,$$

we can write it as

$$\frac{A}{\sqrt{(A^2+B^2)}}x+\frac{B}{\sqrt{(A^2+B^2)}}y+\frac{C}{\sqrt{(A^2+B^2)}}=0.$$

Putting $\dfrac{A}{\sqrt{(A^2+B^2)}}=\cos\theta$, $\dfrac{B}{\sqrt{(A^2+B^2)}}=\sin\theta$, and

$$\frac{C}{\sqrt{(A^2+B^2)}}=-p,$$

we see that it reduces to the form (1), and therefore the length of the perpendicular from the point (x_1, y_1) on the line $Ax+By+C=0$ is

$$x_1\cos\theta+y_1\sin\theta-p,$$

or, substituting for $\cos\theta$, $\sin\theta$ and p,

$$\frac{Ax_1+By_1+C}{\sqrt{(A^2+B^2)}}.$$

If the given point is the origin, this reduces to

$$\frac{C}{\sqrt{(A^2+B^2)}}.$$

Note. The student will find that the value of the expression

$$\frac{Ax_1+By_1+C}{\sqrt{(A^2+B^2)}}$$

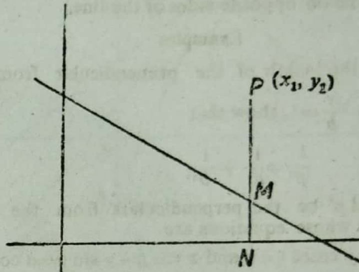
in numerical problems is sometimes positive, sometimes negative. He should disregard the negative sign and give the result as positive unless there is some special reason for not doing so.

3.61 Condition that the point (x_1, y_1) may lie on one side or the other of the straight line $Ax+By+C=0$.

Let P be the point (x_1, y_1).

Let PN be the ordinate through P meeting the given line in M.

PM will be positive for points on one side of the line and negative for points on the other side, for the lengths PM will be oppositely directed in the two cases.



The abscissa of M is x_1 and since M lies on straight line $Ax+By+C=0$, its ordinate y' will be given by

$$Ax_1+By'+C=0.$$

Hence,
$$y'=-\frac{Ax_1+C}{B}$$

Therefore,
$$PM=PN-MN$$
$$=y_1-y'$$
$$=y_1+\frac{Ax_1+C}{B}.$$
$$=\frac{Ax_1+By_1+C}{B}$$

The point (x_1, y_1) *thus lies on one side or the other of the straight line* $Ax+By+C=0$ *according as the expression* Ax_1+By_1+C *is positive or negative.*

If the expression Ax_1+By_1+C *be positive, the point* (x_1, y_1) is said to lie on the positive side of the line $Ax+By+C=0$, and if it be *negative*, the point is said to lie on the *negative side.*

The same point can be on the positive side of a straight line and on the negative side of the same straight line depending upon how the equation to the straight line is written. For example, the origin is on the positive side of the line $Ax+By+C=0$. But if the

equation to the line be written as $-Ax-By-C=0$, the side on which the origin lies, becomes negative. The student should try to explain this.

Corollary. Two points (x_1, y_1) (x_2, y_2) lie on the same side of the line $Ax+By+C=0$, if the expressions Ax_1+By_1+C and Ax_2+By_2+C have the same sign. But if these expressions have opposite signs, the points lie on opposite sides of the line.

Examples

1. If p be the length of the prependicular from the origin on the line $\frac{x}{a}+\frac{y}{b}=1$, show that

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

2. If p and p' be the perpendiculars from the origin upon the straight lines whose equations are

$$x \sec \theta+y \operatorname{cosec} \theta=a \text{ and } x \cos \theta-y \sin \theta=a \cos 2\theta;$$

prove that $4p^2+p'^2=a^2$. (*Rajasthan*, 1963)

3. Prove that the product of the perpendicular from the two points $(\pm\sqrt{a^2-b^2}, 0)$ on the straight line

$$\frac{x}{a} \cos \phi+\frac{y}{b} \sin \phi=1 \text{ is } b^2.$$ (*Calcutta*, 1969)

4. Find the distance between the straight line $x-2y+1=0$ and a parallel to it through the point $(-1, 2)$.

Determine the location of this point with respect to the given line.

Ans. $\frac{4}{\sqrt{5}}$

5. Prove that the triangle formed by joining the points whose coordinates are $(1, 2)$, $(-2, 4)$, $(3, -1)$ lies wholly on the positive side of the line $x+y=1$.

6. Two straight lines drawn through the point $(0, 1)$ are such that the length of the perpendicular dropped from the point $(2, 2)$ on each is 1 unit of length. Find their equations.

What is the equation of the straight line joining the feet of the perpendiculars? *Ans.* $y=1$, $4x-3y+3=0$; $2x+y=5$.

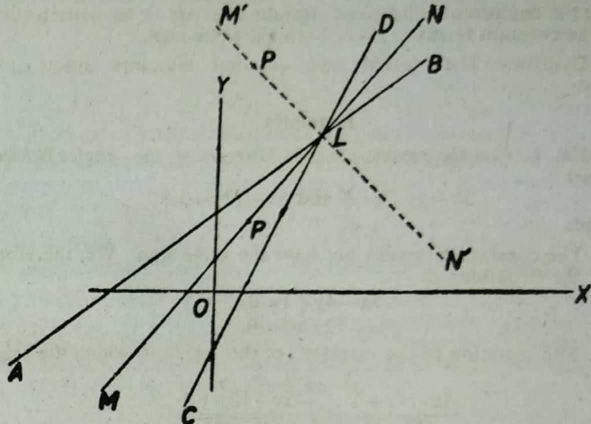
3.7 Straight line bisecting the angles between two given straight lines.

Let the straight lines ALB and CLD be represented by

$$a_1x+b_1y+c_1=0, \quad \ldots(1)$$

and

$$a_2x+b_2y+c_2=0. \quad \ldots(2)$$



Let P be any point (x', y') on the internal bisector MLN or the external bisector $M'LN'$. The lengths of the perpendiculars from P on ALB and CLD will have the same numerical value.

Let the equations (1) and (2) be so written that c_1 and c_2 have the same sign. The origin and the point P lie on the same side of each line provided P moves on LM, the bisector of $\angle ALC$ which contains the origin. If, however, P moves on the other bisector, the point P and the origin lie on the same side of one line and on opposite sides of the other line.

The lengths of perpendiculars from (x', y') on (1) and (2) are respectively

$$\frac{a_1x'+b_1y'+c_1}{\sqrt{a_1^2+b_1^2}} \text{ and } \frac{a_2x'+b_2y'+c_2}{\sqrt{a_2^2+b_2^2}}.$$

Further, since the perpendiculars from the origin upon (1) and (2) will have the same signs as c_1 and c_2 respectively, the locus of P when it lies on the *bisector of the angle containing the origin will be*

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}.$$

The equation to the *bisector of the angle not containing the origin will be*

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}.$$

Note. It must be remembered that before using the above formulae the equation to the given straight lines are to be written such that the constant terms in them have the same sign.

Corollary. The internal and external bisectors are at right angles.

Examples

Ex. 1. *Find the equation to the bisectors of the angles between the lines*

$$3x-4y+1=0 \text{ and } 5x+12y-3=0.$$

Solution.

The constant terms do not have the same sign. We, therefore, write the equations as

$$3x-4y+1=0,$$

and

$$-5x-12y+3=0.$$

The equation to the bisector of the angle in which the origin lies is

$$\frac{3x-4y+1}{\sqrt{3^2+4^2}}=\frac{-5x-12y+3}{\sqrt{5^2+12^2}},$$

or

$$32x+4y-1=0.$$

The equation to the bisector of the other angle is

$$\frac{3x-4y+1}{\sqrt{3^2+4^2}}=-\frac{-5x-12y+3}{\sqrt{5^2+12^2}},$$

or

$$7x-56y+14=0. \text{ i.e., } x-8y+2=0.$$

Ex. 2. Find the equation of the bisector of that angle between the lines

$$4x-3y+1=0 \text{ and } 12x+5y+13=0 \text{ in which the origin lies.}$$

Ans. $2x+16y+13=0$.

Ex. 3. Find the centre of the inscribed circle of the triangle the equations of whose sides are $4y+3x=0$, $12y-5x=0$ and $y-15=0$.

Ans. (1, 8).

3.8 Concurrence of three straight lines.

Let the equations to three given lines be

$$a_1x+b_1y+c_1=0, \quad \text{...(1)}$$

$$a_2x+b_2y+c_2=0, \quad \text{...(2)}$$

and

$$a_3x+b_3y+c_3=0. \quad \text{...(3)}$$

Let (x_1, y_1) be the common point of intersection of the lines.

Then,

$$a_1x_1+b_1y_1+c_1=0,$$

$$a_2x_1+b_2y_1+c_2=0,$$

and

$$a_3x_1+b_3y_1+c_3=0.$$

Eliminating x_1 and y_1,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0,$$

which is the condition for the concurrence of the lines.

There is yet another test, very effective at times, to study the concurrence of three given lines. *If three constants λ, μ, ν can be so determined that*

$$\lambda(a_1x+b_1y+c_1)+\mu(a_2x+b_2y+c_2)+\nu(a_3x+b_3y+c_3) \quad \text{... (4)}$$

vanish identically, the lines (1), (2) and (3) will meet in a point.

The coordinates of the point of intersection of any two of the given lines, say (2) and (3), will make the expressions $a_2x+b_2y+c_2$ and $a_3x+b_3y+c_3$ separately zero. Since the expression (4) vanishes for all values of x and y, the coordinates of the point of intersection of (2) and (3) will also satisfy the equation

$$a_1x+b_1y+c_1=0$$

i.e., the three lines will be concurrent.

Examples

Ex. 1. Show that the lines $2x+y-1=0$, $4x+3y-3=0$ and $3x+2y-2=0$ are concurrent.

Ex. 2. Find the value of k for which the lines

$$3x-4y+5=0,\ 7x-8y+5=0 \text{ and } 4x+5y+k=0$$

are concurrent.

Ans. -45.

Ex. 3. In any triangle prove the concurrence of the following :

(1) The medians.
(2) The altitudes.
(3) The bisectors of the angles, either all internal or one internal and two external.
(4) The perpendicular bisectors of the sides.

Solution. (1) Let the sides of the triangle be

$$u_1 \equiv x\cos\alpha_1+y\sin\alpha_1-p_1=0,$$

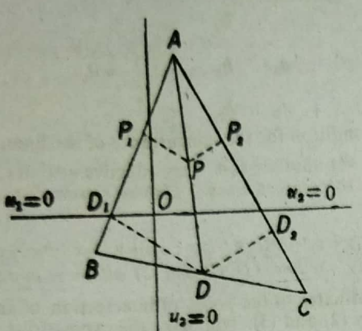
$$u_2 \equiv x\cos\alpha_2+y\sin\alpha_2-p_2=0,$$

and

$$u_3 \equiv x\cos\alpha_3+y\sin\alpha_3-p_3=0.$$

Let P be any point (x, y) on the median through the intersection of $u_1=0$, $u_2=0$.

The perpendicular PP_1 is u_1; the perpendicular PP_2 is u_2.



But $$\frac{PP_1}{PP_2}=\frac{DD_1}{DD_2}=\frac{\sin B}{\sin C},$$

where DD_1 and DD_2 are perpendiculars from D upon $u_1=0$ and $u_3=0$.

The locus of P or the equation to AD is therefore

$$u_1 \sin C-u_3 \sin B=0. \quad \text{.. (1)}$$

The equations to the other two medians are similarly

$$u_2 \sin B-u_3 \sin A=0. \quad \text{...(2)}$$

and $$u_3 \sin A-u_1 \sin C=0. \quad \text{...(3)}$$

Adding (1), (2) and (3) the left-hand side vanishes identically. Hence the medians are concurrent.

The student should establish the other result in a like manner.

Ex 4. The eqations to the sides of a triangle (in rectangular cartesian coordinates) are

$$p_r \equiv a_r x+b_r y+c_r=0, \; r=1, 2, 3.$$

Show that the coordinates of the orthocentre of the triangle satisfy the equation

$$\lambda_1 p_1=\lambda_2 p_2=\lambda_3 p_3$$

where

$$\lambda_1=a_2a_3+b_2b_3, \quad \lambda_2=a_3a_1+b_3b_1$$

and $$\lambda_3=a_1a_2+b_1b_2$$ (*Lucknow*, 1952)

3.9 Equation to a straight line in polar coordinates.

The general equation to a straight line in cartesian coordinates is $px+qy=1$. Converting this into polar coordinates,

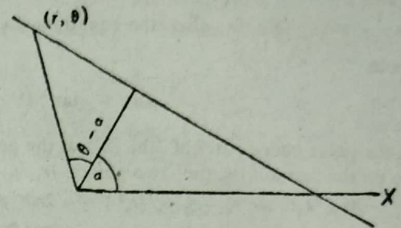
$$pr \cos \theta+qr \sin \theta=1,$$

or $$\frac{1}{r}=p \cos \theta+q \sin \theta.$$

which, in polar coordinates, is the most general equation to a straight line.

We shall obtain from first principle the equation to a straight line in the following cases.

Case I. If p is the length of the perpendicular from the pole upon a straight line, α the angle which this perpendicular makes

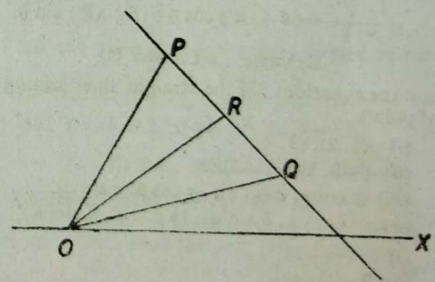


with initial line and (r, θ) the coordinates of any movable point on the line, then obviously,

$$p=r \cos (\theta-\alpha),$$

which is the equation to the given line.

Case II. Let us obtain the equation to a straight line joining two given points.



Let Q and R be the two given points with coordinates (r_1, θ_1) and (r_2, θ_2) on the line QR. Let P be any point on this line.

Since $$\Delta OQP=\Delta OQR+\Delta ORP,$$

$$\tfrac{1}{2} rr_1 \sin (\theta-\theta_1)=\tfrac{1}{2} r_1r_2 \sin (\theta_2-\theta_1)+\tfrac{1}{2} rr_2 \sin (\theta-\theta_2),$$

or $$r_1r_2 \sin (\theta_2-\theta_1)+rr_2 \sin (\theta-\theta_2)+rr_1 \sin (\theta_1-\theta)=0,$$

i.e., $$\frac{\sin (\theta_2-\theta_1)}{r}+\frac{\sin (\theta-\theta_2)}{r_1}+\frac{\sin (\theta_1-\theta)}{r_2}=0.$$

which is the required equation.

Examples

1. Show that the equation to any straight line passing through the pole and making an angle α with the initial line is $\theta=\alpha$.

2. Show that the equation to any straight line perpendicular to the line $\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta$ is $\frac{l'}{r}=-\sin(\theta-\alpha)-e\sin\theta$. Obtain also the equation to the parallel through the pole.

Ans. $\theta=\tan^{-1}\left(-\frac{\cos\alpha+e}{\sin\alpha}\right)$.

3. Find the polar coordinates of the foot of the perpendicular from the pole on the line joining the two points (r_1, θ_1) and (r_2, θ_2).

Ans. $r_1r_2\sin(\theta_1-\theta_2)/\sqrt{\{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)\}}$,

$\tan^{-1}\frac{r_2\cos\theta_2-r_1\cos\theta_1}{r_1\sin\theta_1-r_2\sin\theta_2}$.

4. Show that the equation to any straight line passing through the point of intersection of the lines

$$\frac{l}{r}=A\cos\theta+B\sin\theta \quad\text{and}\quad \frac{l'}{r}=A'\cos\theta+B'\sin\theta$$

is

$$\frac{l+\lambda l'}{r}=(A+\lambda A')\cos\theta+(B+\lambda B')\sin\theta.$$

Examples on Chapter III

1. Find the equations of the straight lines joining the following pairs of points :

(i) $(1, 2)$, $(3, 4)$.
(ii) $(-3, 2)$ $(3, -2)$.
(iii) $(c\cos\alpha, c\sin\alpha)$, $(c\cos\beta, c\sin\beta)$.
(iv) $(ca\cos\phi, b\sin\phi)$, $(a\cos\phi', b\sin\phi')$.

Ans. (i) $y=x+1$.
(ii) $2x+3y=0$.
(iii) $x\cos\frac{\alpha+\beta}{2}+y\sin\frac{\alpha+\beta}{2}=c\cos\frac{\alpha-\beta}{2}$.
(iv) $\frac{x}{a}\cos\frac{\phi+\phi'}{2}+\frac{y}{b}\sin\frac{\phi+\phi'}{2}=\cos\frac{\phi-\phi'}{2}$.

2. If the length of the perpendicular from the origin on the line $y=mx+c$ be equal to a, show that

$$c=\pm a\sqrt{1+m^2}.$$

3. Find the equation of the line through the point $(2, 3)$ which makes equal intercepts on the axes.

Ans. $x+y=5$.

4. Show that the perpendicular from the origin upon the straight line joining the points $(c\cos\alpha, c\sin\alpha)$ and $(c\cos\beta, c\sin\beta)$ bisects the distance between them.

Hint. The points are equidistant from the origin.

5. Find the angles between the following pairs of straight lines :

(i) $x+y\sqrt{3}=4$, $x\sqrt{3}-y=5$.
(ii) $2x+3y-1=0$, $3x-5y+3=0$.
(iii) $y=(2+\sqrt{5})x+1$, $y+(2-\sqrt{5})x=2$.
(iv) $x\sqrt{2}+y\sqrt{3}=2$, $x\sqrt{3}+y\sqrt{2}=5$.

Ans. (i) $90°$, (ii) $\tan^{-1}\frac{19}{9}$, (iii) $\tan^{-1}2$, (iv) $\tan^{-1}\frac{1}{2\sqrt{6}}$.

6. Find the equation of the locus of a point which moves in a plane so that the sum of squares of its distance from the two lines $7x-4y-10=0$ and $4x+7y+5=0$ is always equal to 3.

(*I. I. T. Admission*, 1963)

Ans. $13(x^2+y^2)-20x+30y-14=0$.

7. Find the equations to the lines through the point of intersection of $5x-y-3=0$ and $3x+7y-17=0$ which are at a distance of 2 units from the origin.

Ans. $y=2$, $3y+4x=10$.

8. Find the areas of the triangles formed by the lines

(i) $y=x$, $y=2x$ and $y=3x+4$.
(ii) $y=x+1$, $x=2y-3$, $3x-4y+7=0$.
(iii) $3x+4y=2$, $x+2y=3$, $3x+y=3$. (*Roorkee*, 1966)

Ans. (i) 4 sq. units, (ii) 1 sq. unit, (iii) $\frac{529}{180}$ sq. units.

9. A straight line drawn through the point $(2, 1)$ is such that the point of intersection with the line $y-2x+6=0$ is at a distance $3\sqrt{2}$ from this point. Find the direction of the line.

Ans. $45°$.

10. Prove that the diagonals of the parallelogram formed by the four straight lines $\sqrt{3}x+y=0$, $\sqrt{3}y+x=0$, $\sqrt{3}x+y=1$, $\sqrt{3}y+x=1$ are at right angles to one another.

(*I. I. T. Admission*, 1962)

11. The base BC of a triangle ABC is bisected at the point (p, q) and the equations to the sides AB and AC are $px+qy=1$ and $qx+py=1$. Show that the equation to the median through A is

$$(2pq-1)(px+qy-1)=(p^2+q^2-1)(qx+py-1).$$

12. If the three straight lines

$$p_1x+q_1y=1,\ p_2x+q_2y=1 \text{ and } p_3x+q_3y=1$$

all pass through one point, show that the three points (p_1, q_1), (p_2, q_2) and (p_3, q_3) will lie in one straight line.

13. The three lines $x+2y+3=0$, $x+2y-7=0$, $2x-y-4=0$ form the three sides of two squares. Find the equations to the fourth side of each square.

Ans. $y-2x=6$, $y-2x+14=0$.

14. A straight line moves so that the sum of the reciprocals of its intercepts on two fixed intersecting lines is constant; show that it passes through a fixed point.

15. If a line be such that the sum of perpendiculars let fall on it from a number of fixed points each multiplied by a constant be zero; show that it will pass through a fixed point.

16. Find the locus of the foot of the perpendicular from the origin upon the line joining the points $(a \cos \theta, b \sin \theta)$ and $(-a \sin \theta, b \cos \theta)$, θ varying in any manner.

Ans. $2(x^2+y^2)^2=a^2x^2+b^2y^2$.

17. Find the coordinates of the orthocentre of the triangle whose vertices are $(0, 0)$, $(2, -1)$ and $(-1, 3)$.

(I. I. T. Admission, 1974)

Ans. $(-4, -3)$.

18. If the vertices of a triangle have integral coordinates, prove that the triangle can not be equilateral.

(I. I. T. Admission, 1975)

19. Find the equation of straight lines passing through $(-2, -3)$ and having an intercept of length 3 units between the lines $4x+3y=12$ and $4x+3y=3$. *(I. I. T. Admission, 1977)*

Ans. $x+2=0$, $7x+24y+86=0$.

20. A variable straight line, drawn through the point of intersection of the lines $x/a+y/b=1$ and $x/b+y/a=1$, meets the coordinate axes in A, B. Show that the locus of the middle point of AB is $2xy(a+b)=ab(x+y)$. *(I. I. T. Admission, 1977)*

21. Two vertices of a triangle are $(5, -1)$ and $(-2, 3)$. If the orthocentre of the triangle lies at the origin, find the coordinates of the third point. *(I. I. T. Admission, 1979)*

Ans. $\left(\dfrac{28}{29}, \dfrac{49}{29}\right)$.

22. Verify that the lines $x-y-7=0$, $x+2y+8=0$ and $2x+y+1=0$ pass through a common point, and that this point is equidistant from the three points $(5, -4)$ $(3, -2)$ and $(-1, -6)$.

(Calcutta, 1956)

23. Find the locus of a point which moves such that the square of its distance from the base of an isosceles triangle is equal to the rectangle under its distances from the other sides.

Hint. Take the base and the perpendicular from the vertex as coordinate axes.

24. A variable line cuts off from n given concurrent straight lines intercepts, the sum of the reciprocals of which is constant. Show that it always passes through a fixed point.

Hint. Use polar coordinates.

25. Show that the line $lx+my+n=0$ bisects the angle between $px+qy+r=0$, and

$$(px+qy+r)(l^2+m^2)=2(lp+mq)(lx+my+n).$$

26. Find the area of parallelogram the equations to whose sides are

$$l_1x+m_1y+n_1=0, \qquad l_2x+m_2y+n_2=0,$$
$$l_1x+m_1y+n_3=0. \qquad l_2x+m_2y+n_4=0.$$

Ans. $\dfrac{(n_3-n_1)(n_4-n_2)}{l_1m_2-m_1l_2}$.

27. If the coordinates of the vertices of a triangle ABC are $(0, 3\sqrt{3}-2)$, $(-2\sqrt{3}, -2)$ and $(3, -2)$ respectively, find (*i*) coordinates of the orthocentre, (*ii*) equation to the bisector of the angle ACB, and (*iii*) coordinates of the centre of the circle that touches CA and CB, and has a radius $2+\sqrt{3}$.

Ans. (*i*) $(0, 0)$; (*ii*) $x+y\sqrt{3}=3-2\sqrt{3}$; (*iii*) $(-2\sqrt{3}, \sqrt{3})$, $\left(4+\dfrac{2}{\sqrt{3}}, \sqrt{3}\right)$, $(6+2-\sqrt{3}, 4-\sqrt{3})$, or $\left(2-\dfrac{2}{\sqrt{3}}, -4-\sqrt{3}\right)$.

28. Given n straight lines and a fixed point O; through O is drawn a straight line meeting these lines in the points R_1, R_2, $R_3 \ldots R_n$, and on it is taken a point R such that

$$\frac{n}{OR}=\sum_{p=1}^{n}\frac{1}{OR_p},$$

show that the locus of R is a straight line.

29. The cartesian equations of the sides BC, CA, AB of a triangle are

$$u_1 \equiv a_1x+b_1y+c_1=0,$$
$$u_2 \equiv a_2x+b_2y+c_2=0,$$
$$u_3 \equiv a_3x+b_3y+c_3=0.$$

and a line is drawn through A parallel to BC; prove that its equation is

$$\frac{u_3}{a_3b_1-a_1b_3}-\frac{u_2}{a_1b_2-a_2b_1}=0.$$

Show also that the equation of the line through A bisecting the side BC is

$$\frac{u_3}{a_3b_1-a_1b_3}-\frac{u_2}{a_1b_2-a_2b_1}=0.$$

30. Show that the centroids of the triangles of which the altitudes lie along the lines

$$y-m_1x=0, \qquad y-m_2x=0, \qquad y-m_3x=0$$

lie on $y\,(3+m_2m_3+m_3m_1+m_1m_2)=x\,(m_1+m_2+m_3+3m_1m_2m_3)$.

(*Andhra*, 1961)

31. Show that the area of the triangle formed by the lines $y=m_rx+c_r$, $r=1, 2, 3$ is

$$\frac{1}{2}\left\{\frac{(c_1-c_2)^2}{m_1-m_2}+\frac{(c_2-c_3)^2}{m_2-m_3}+\frac{(c_3-c_1)^2}{m_3-m_1}\right\}.$$

32. Discuss the loci represented by the following equations, and draw their graphs :

(i) $y=|x|$.

(ii) $|x|+|y|=1$.

(iii) $|x|+2|y|<1$. (*I. A. S.*, 1977)

Hint. $|x|=x$ if $x\geqslant 0$, whereas $|x|=-x$ if $x<0$.

Ans. (i) Two straight lines intersecting at the origin and lying in the first and second quadrants.

(ii) The sides of a square.

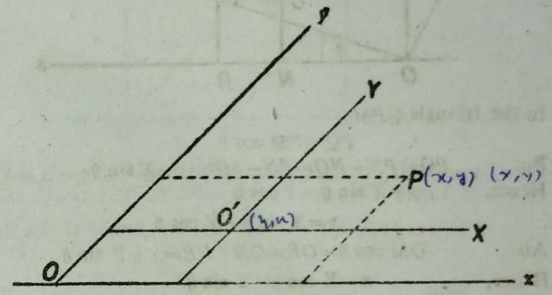
(iii) The interior of a rhombus.

CHAPTER IV

CHANGE OF AXES, INVARIANTS

4.1 Change of Origin.

Let (x, y) be the coordinates of any point P referred to axes Ox and Oy. Let O' be the point (h, k) and $O'X$ and $O'Y$ lines drawn



parallel to Ox and Oy respectively Let (X, Y) be the coordinates of P referred to $O'X$ and $O'Y$ as coordinate axes. If the straight lines are drawn parallel to these axes, it is easily seen that

$$x=X+h, \qquad y=Y+k.$$

If the locus of P with respect to Ox and Oy be $f(x, y)=0$, the equation to the same locus when the origin is transferred to O' the axes retaining their directions, becomes $f(X+h, Y+k)=0$ where X, Y are the current coordinates with reference to the new axes.

Note. The above law of transformation is independent of the angle between the axes.

It may easily be seen that this transformation affects only the first degree terms in an equation of the second degree.

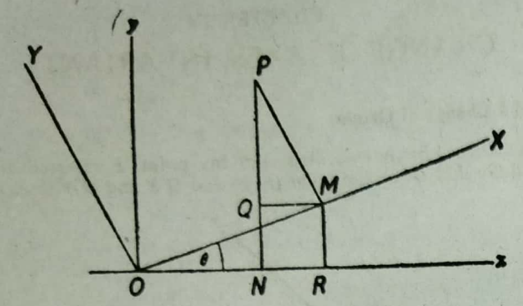
4.2 Change of rectangular axes without change of origin.

Let Ox, Oy be changed to the new axes OX, OY where θ is the angle xOX or yOY. Let the coördinates of any point P be (x, y) referred to old axes and (X, Y) referred to new axes.

Draw PN, PM perpendicular to Ox and OX, and MR, MQ perpendicular to Ox and PN.

Then $\qquad ON=x,\ NP=y,\ OM=X,\ MP=Y.$

Also $\qquad \angle QPM=\theta.$



In the triangle QPM,

$$PQ=PM \cos \theta.$$

But $PQ=PN-NQ=PN-MR=y-X \sin \theta.$

Hence, $y-X \sin \theta=Y \cos \theta$

i. e., $$\mathbf{y=X \sin \theta+Y \cos \theta.} \qquad \ldots(1)$$

Also, $OM \cos \theta=OR=ON+NR=x+Y \sin \theta.$

Hence, $$\mathbf{x=X \cos \theta-Y \sin \theta.} \qquad \ldots(2)$$

From (1) and (2), or independently,

$$X=x \cos \theta + y \sin \theta$$

$$Y=-x \sin \theta+y \cos \theta.$$

Note. The above relations can easily be remembered with the help of the formula $x+iy=(X+iY)(\cos \theta+i \sin \theta)$,

$$f(X \cos \theta+Y \sin \theta,\ X \sin \theta+Y \cos \theta)=0$$

when the axes are rotated through an angle θ without change of origin.

If the origin be transferred to (h, k) and the axes turned through an angle θ, the transformation will be given by the relations

$$x=h+X \cos \theta-Y \sin \theta,\ y=k+X \sin \theta+Y \cos \theta.$$

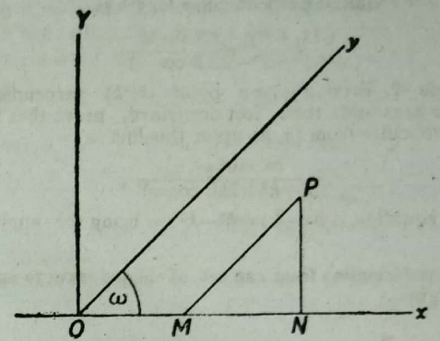
Example. The equation

$$14x^2-4xy+11y^2=60$$

represents a certain locus. Show that if the axes be rotated through an angle $\tan^{-1} 2$, without change of origin, the equation to the locus referred to new axes becomes

$$\frac{X^2}{6}+\frac{Y^2}{4}=1.$$

4.3 Transformation of oblique axes to rectangular axes with the same origin and same x-axis.



Let the coordinates of a point P be (x, y) referred to oblique axes Ox, Oy and (X, Y) referred to rectangular axes OX, OY.

If PM is drawn parallel to Oy and PN parallel to OY, then obviously,

$$X=x+Y \cot \omega$$

i.e., $$x=X-Y \cot \omega,$$

and $$y=Y \operatorname{cosec} \omega.$$

Example 1. *Obtain the length of the perpendicular from (x_1, y_1) upon the line $Ax+By+C=0$, the axes being inclined at an angle ω.*

Solution. Transforming from oblique axes to rectangular axes with the same origin and same x-axis, the equation to the given line becomes

$$A(X-Y \cot \omega)+BY \operatorname{cosec} \omega+C=0$$

or $$Ax \sin \omega+Y(B-A \cos \omega)+C \sin \omega=0.$$

The perpendicular from (x_1', y_1') on this is

$$\frac{A \sin \omega\, x_1'+(B-A \cos \omega)\, y_1'+C \sin \omega}{\sqrt{[A^2 \sin^2 \omega+(B-A \cos \omega)^2]}}. \qquad \ldots(1)$$

Since (x_1', y_1') are the coordinates of (x_1, y_1) referred to new axes,

$$x_1=x_1'-y_1' \cot \omega$$

and $$y_1=y_1' \operatorname{cosec} \omega,$$

i.e., $$x_1'=x_1+y_1 \cos \omega \text{ and } y_1'=y_1 \sin \omega.$$

Substituting in (1), the numerator becomes

$$A \sin \omega\,(x_1+y_1 \cos \omega)+(B-A \cos \omega)\, y_1 \sin \omega+C \sin \omega,$$

i.e., $$(Ax_1+By_1+C) \sin \omega.$$

Hence the required perpendicular length is

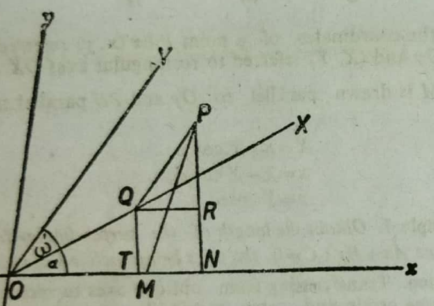
$$\frac{(Ax_1+By_1+C)\sin\omega}{\sqrt{(A^2+B^2-2AB\cos\omega)}}.$$

Example 2. From a given point (h, k) perpendiculars are drawn to the axes and their feet are joined, prove that the length of the perpendicular from (h, k) upon this line is

$$\frac{hk\sin^2\omega}{\sqrt{(h^2+k^2+2hk\cos\omega)}},$$

and that its equation is $hx-ky=h^2-k^2$, ω being the angle between the axes.

4.4 Transformation from one set of oblique axes to another with the same origin.



Let Ox, Oy be a set of oblique axes containing an angle ω and let OX, OY be another set containing an angle ω'. Let the coordinates of a point P be (x, y) and (X, Y) referred to the first and the second set respectively. Let the angle xOX be α.

It is required to find the law of transformation from (x, y) *to* (X, Y).

Draw PM and PQ parallel to Oy and OY respectively. Then, $OM=x$, $OQ=X$, $PM=y$, $PQ=Y$.

Let PN, QT be drawn perpendiculars to Ox and QR perpendicular to PN.

In the triangles PMN, PQR, $\angle PMN=\omega$, $\angle PQR=\omega'+\alpha$.

Therefore, $y\sin\omega=PN=PR+RN=PR+QT$

$$=Y\sin(\omega'+\alpha)+X\sin\alpha,$$

i.e., $$y=X\frac{\sin\alpha}{\sin\omega}+Y\frac{\sin(\alpha+\omega')}{\sin\omega}.$$

Further, since the inclinations of OX and OY to Oy are $\omega-\alpha$ and $\omega-(\alpha+\omega')$, we obtain as above

$$x\sin\omega=X\sin(\omega-\alpha)+Y\sin\{\omega-(\alpha+\omega')\}$$

i.e., $$x=X\frac{\sin(\omega-\alpha)}{\sin\omega}+Y\frac{\sin\{\omega-(\alpha+\omega')\}}{\sin\omega}.$$

The equation $f(x, y)=0$ *thus transforms into*

$$f\left(X\frac{\sin(\omega-\alpha)}{\sin\omega}+Y\frac{\sin(\omega-\beta)}{\sin\omega},\ X\frac{\sin\alpha}{\sin\omega}+Y\frac{\sin\beta}{\sin\omega}\right)=0,$$

where α *and* β *are the inclinations of the new axes of x and y to the old x-axis.*

Note The equations of transformation in the case of rectangular and oblique axes, the origin remaining the same, have the same form, viz.,

$$x=\lambda X+\mu Y,\quad y=\lambda' Y+\mu' Y,$$

where λ, μ, λ', μ' are constants depending upon the particular axes chosen.

Corollary 1. *The relation between the old and new coordinates in a general transformation of axes is of the form*

$$x=pX+qY+r \text{ and } y=p'X+q'Y+r'.$$

Corollary 2. *The transformation of coordinate axes does not affect the degree of an equation.*

4.5 Removal of xy term from $ax^2+2hxy+by^2$, the axes being rectangular.

Since a change of origin alone does not affect the highest degree terms in an equation, we shall assume the origin to be fixed and rotate the axes through an angle θ. If X, Y are the new coordinates,

$$x=X\cos\theta-Y\sin\theta$$

and $$y=X\sin\theta+Y\cos\theta.$$

Substituting in the expression $ax^2+2hxy+by^2$, we get

$$a(X\cos\theta-Y\sin\theta)^2+2h(X\cos\theta-Y\sin\theta)(X\sin\theta+Y\cos\theta)$$
$$+b(X\sin\theta+Y\cos\theta)^2,$$

which can be written as

$$(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)X^2$$
$$-2\{(a-b)\sin\theta\cos\theta-h(\cos^2\theta-\sin^2\theta)\}XY$$
$$+(a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta)Y^2.$$

The term in XY disappears if

$$(a-b)\sin\theta\cos\theta-h\cos 2\theta=0,$$

i.e., if $$\tan 2\theta=\frac{2h}{a-b}.$$

Since a real value of θ is always possible, the axes can be so chosen as to reduce the expression $ax^2+2hxy+by^2$ to the form AX^2+BY^2.

If the expression $ax^2+2hxy+by^2$ is a perfect square, *i.e.*, if $h^2=ab$, rotation through the above angle will make either the coefficient of $X^2=0$, or the coefficient of $Y^2=0$. The proof is left as an exercise to the reader.

Example. Find the angle through which the axes must be rotated so that the expression $x^2+2\sqrt{3}xy-y^2$ may change into an expression in which the XY term is missing.

Ans. 30°.

4.6 Invariants. If any change of axes transforms $ax^2+2hxy+by^2$ into $a'P^2+2h'XY+b'Y^2$, then

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'}$$

and
$$\frac{ab-h^2}{\sin^2\omega}=\frac{a'b'-h'^2}{\sin^2\omega'},$$

where ω and ω' are the angles of inclination of the old and new sets of axes.

Obviously, the origin will be same for the two sets of axes. Let (x, y) be the coordinates of any point P referred to old axes and (X, Y) the coordinates of the same point referred to new axes. Since the distance OP is independent of the choice of axes, $x^2+y^2+2xy\cos\omega$ is changed into $X^2+Y^2+2XY\cos\omega'$.

Also, by supposition, $ax^2+2hxy+by^2$ is changed into $a'X^2+2h'XY+b'Y^2$.

Therefore, $ax^2+2hxy+by^2+\lambda\,(x^2+y^2+2xy\cos\omega)$ will be changed into

$$a'X^2+2h'XY+b'Y^2+\lambda\,(X^2+Y^2+2XY\cos\omega'),$$

where λ is any constant.

Since the transformation relation is of the form $x=pX+qY$, $y=p'X+q'Y$, any value of λ which makes one of the above expressions a perfect square will also make the other a perfect square.*

The first expression is a perfect square if

$$(a+\lambda)\,(b+\lambda)-(h+\lambda\cos\omega)^2=0 \qquad \text{...(1)}$$

and the second is a perfect square if

$$(a'+\lambda)\,(b'+\lambda)-(h'+\lambda\cos\omega')^2=0 \qquad \text{...(2)}$$

*If the first expression as a perfect square is $(Px+QY)^2$, it will be changed into $\{P\,(pX+qY)+Q\,(p'X+q'Y)\}^2$ *i.e.*, into $(P'X+Q'Y)^2$ which is also a perfect square.

Equations (1) and (2) in λ must be the same.

Rearranging terms in (1) and (2),

$$\lambda^2\sin^2\omega+(a+b-2h\cos\omega)\,\lambda+ab-h^2=0 \qquad \text{...(3)}$$

and
$$\lambda^2\sin^2\omega'+(a'+b'-2h'\cos\omega')\,\lambda+a'b'-h'^2=0. \qquad \text{...(4)}$$

Comparing coefficients in (3) and (4), we obtain

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'} \qquad \text{...(5)}$$

and
$$\frac{ab-h^2}{\sin^2\omega}=\frac{a'b'-h'^2}{\sin^2\omega'}. \qquad \text{...(6)}$$

The expressions $\dfrac{a+b-2h\cos\omega}{\sin^2\omega}$ and $\dfrac{ab-h^2}{\sin^2\omega}$ are called **invariants**, for their values are independent of the transformation of axes.

Corollary. *If any change of rectangular axes converts $ax^2+2hxy+by^2$ into $a'X^2+2h'XY+b'Y^2$, then*

$$a+b=a'+b',$$

and
$$ab-h^2=a'b'-h'^2.$$

This follows from (5) and (6) on putting $\omega'=\omega=\pi/2$.

Examples on Chapter IV

1. Transform the equation

$$x^2-3y^2+4x+6y=0$$

by transferring the origin to the point $(-2, 1)$, the coordinate axes remaining parallel. (*Bihar*, 1968)

Ans. $x^2-3y^2=1$.

2. Show that the equation

$$12x^2-10xy+2y^2+11x+5y+2=0$$

can be made a homogeneous equation of second degree by transferring the origin to a properly chosen point. (*Ranchi*, 1968)

3. The coordinates of a point referred to two sets of rectangular axes with the same origin are (x, y) and (X, Y). If $ux+vy$, where u and v are independent of x and y, becomes $UX+VY$, show that

$$u^2+v^2=U^2+V^2.$$

4. If one system of oblique axes be transformed to another system according to the formula

$$x=mx'+ny',\quad y=m'x'+n'y',$$

the origin being the same, then prove that

$$nn'\,(m^2+m'^2-1)=mm'\,(n^2+n'^2-1).$$

5. Use transformation of coordinate axes (two dimensional and rectangular) to establish the following relations between algebra and geometry :

(a) Every first degree equation represents a straight line.
(b) The equation of any line is of first degree.

Hint. Transform the axes so that the coefficient of x or y is absent.

6. Show that if the origin be transferred to (0, 1) and the axes rotated through 45°, the equation

$$5x^2-2xy+5y^2+2x-10y-7=0,$$

referred to new axes becomes

$$\frac{X^2}{3}+\frac{Y^2}{2}=1.$$

7. Prove that the value of g^2+f^2 in the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

remains unaffected by orthogonal transformation without change of origin.

Hint. The first degree terms in the transformed equation are $2(g\cos\theta+f\sin\theta)X+2(f\cos\theta-g\sin\theta)Y$. Square and add the coefficients of X and Y.

8. Prove that the transformation which converts

$$\frac{X^2}{p}+\frac{Y^2}{q} \text{ into } ax^2+2hxy+by^2$$

will convert $\dfrac{X^2}{p-\lambda}+\dfrac{Y^2}{q-\lambda}$ into

$$\frac{ax^2+2hxy+by^2-\lambda(ab-h^2)(x^2+y^2)}{1-\lambda(a+b)+\lambda^2(ab-h^2)}. \quad \textit{(Lucknow 1964)}$$

Hint. $\dfrac{1}{p}+\dfrac{1}{q}=a+b$, and $\dfrac{1}{pq}=ab-h^2$, by invariants.

Now $\dfrac{X^2}{p-\lambda}+\dfrac{Y^2}{q-\lambda}=\dfrac{qX^2-pY^2-\lambda(X^2+Y^2)}{pq-\lambda(p+q)+\lambda^2}$.

Divide numerator and denominator by pq; substitute for $(X^2/p+Y^2/q)$, $(1/pq)$ and $(1/p+1/q)$ and obtain the result.

CHAPTER V

PAIRS OF STRAIGHT LINES

5.1 General Equation.

We have seen in Chapter III that the general equation of the first degree in x and y, *viz.*, $Ax+By+C=0$, represents a straight line. Let us now consider the product equation

$$(Ax+By+C)(A'x+B'y+C')=0 \qquad ...(1)$$

and examine what locus is represented by it.

Since the two factors on the left-hand side of equation (1) are $Ax+By+C$ and $A'x+B'y+C'$, the equation is obviously satisfied if either of these factors is equal to zero. Equation (1) accordingly represents the *pair of straight lines*

$$Ax+By+C=0 \text{ and } A'x+B'y+C'=0.$$

If we multiply the two factors on the left-hand side of (1), we get an equation of the form

$$ax^2+2hxy+by^2+2gx+fy+c=0. \qquad ...(2)$$

where the values of a, b, c, f, g, h are easily determined in terms of A, B, C, A', B', C'. This is the *most general equation* of the second degree and will represent a pair of straight lines provided the coefficient of various powers of x and y and the constant term are suitably determined. The obvious condition is that the expression on the left-hand side of equation (2) should break up into linear factors in x and y.

5.11 Pair of straight lines through the origin. We shall show that the *homogeneous equation of the second degree*

$$ax^2+2hxy+by^2=0$$

represents a pair of straight lines passing through the origin.

Solving the above equation for x*, we obtain

$$x=\frac{-h\pm\sqrt{h^2-ab}}{a}y,$$

i.e., $\quad ax+(h+\sqrt{h^2-ab})\,y=0,$

or $\quad ax+(h-\sqrt{h^2-ab})\,y=0.$

Each of the above is the equation of a straight line passing through the origin.

*This presumes that $a\neq0$. If $a=0$, the result follows immediately.

Hence the homogeneous equation

$$ax^2+2hxy+by^2=0$$

represents two straight lines passing through the origin.

The lines are real and distinct, coincident, or imaginary according as $h^2 >, =$ or $< ab$.

5.12 Angle between the lines $ax^2+2hxy+by^2=0$.

Let the two lines represented by

$$ax^2+2hxy+by^2=0 \qquad ...(1)$$

be $y=m_1x$ and $y=m_2x$.

Then

$$(y-m_1x)(y-m_2x)=0$$

or $$m_1m_2x^2-(m_1+m_2)\,xy+y^2=0$$

is the same as (1).

Comparing coefficients,

$$\frac{m_1m_2}{a}=-\frac{m_1+m_2}{2h}=\frac{1}{b}.$$

This gives

$$m_1m_2=\frac{a}{b},$$

and $$m_1+m_2=-\frac{2h}{b}.$$

If θ be the angle between the given straight lines,

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{\sqrt{\{(m_1+m_2)^2-4m_1m_2\}}}{1+m_1m_2}$$

$$=\frac{\sqrt{4h^2-4ab}}{a+b}.$$

Hence,

$$\theta=\tan^{-1}\left\{\frac{2\sqrt{h^2-ab}}{a+b}\right\}.$$

If $a+b=0$, the lines are perpendicular, and if $h^2=ab$, the lines are coincident.

5.13 Angle between the pair of lines $ax^2+2hxy+by^2=0$, the axes being inclined at an angle ω.

Let the equation to the lines become

$$a'x^2+2h'xy+b'y^2=0$$

when transformed to rectangular axes.

If θ be the angle between the lines,

$$\tan\theta=\frac{2\sqrt{h'^2-a'b'}}{a'+b'}.$$

By invariants,

$$\frac{h^2-ab}{\sin^2\omega}=\frac{h'^2-a'b'}{\sin^2\pi/2}$$

and $$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{a'+b'-2h'\cos\pi/2}{\sin^2\pi/2}$$

i.e., $$h'^2-a'b'=\frac{h^2-ab}{\sin^2\omega},\ \text{and}\ a'+b'=\frac{a+b-2h\cos\omega}{\sin^2\omega}.$$

Hence, $$\tan\theta=\frac{2\sin\omega\sqrt{h^2-ab}}{a+b-2h\sin\omega}.$$

If the lines are perpendicular,

$$a+b-2h\cos\omega=0.$$

5.2 Homogeneous equation of the n^{th} degree.

Let

$$ay^n+by^{n-1}x+cy^{n-2}x^2+......kx^n=0 \qquad ...(1)$$

be a homogeneous equation of n^{th} degree. We shall show that it represents n straight lines each of which passes through the origin.

Since each term in (1) is of degree n, dividing by x^n, we get

$$a\left(\frac{y}{x}\right)^n+b\left(\frac{y}{x}\right)^{n-1}+c\left(\frac{y}{x}\right)^{n-2}+...+k=0.$$

This is an equation of the n^{th} degree in $\frac{y}{x}$, and will have n roots. Denoting the roots by $m_1, m_2......m_n$, the equation can be written as

$$a\left(\frac{y}{x}-m_1\right)\left(\frac{y}{x}-m_2\right)....\left(\frac{y}{x}-m_n\right)=0,$$

or $$a\,(y-m_1x)\,(y-m_2x)...(y-m_nx)=0.$$

Equation (1) is therefore the locus of points lying on one or the other of the n straight lines

$$y-m_1x=0,\ y-m_2x=0,......,\ y-m_nx=0,$$

each of which passes through the origin.

Note. The n straight lines are not necessarily real and distinct. This follows from the theory of equations.

Examples

1. If the distance of a given point (p, q) from each of two straight lines through the origin is k, prove that the equation of the straight lines is

$$(py-qx)^2=k^2(x^2+y^2).$$ *(Roorkee Ent., 1979)*

2. Prove that the equation to the straight lines, each of which makes an angle α with the line $y=x$, is

$$x^2-2xy\sec 2\alpha+y^2=0.$$ *(Calcutta, 1969)*

3. Show that the two straight lines

$$x^2(\tan^2\theta+\cos^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0$$

make with the axes of x angles α, β such that

$$\tan\alpha\sim\tan\beta=2.$$ *(Delhi, 1969)*

4. Find the equation to the pair of straight lines perpendicular to the pair given by

$$ax^2+2hxy+by^2=0.$$

Solution. If $y=m_1x$, $y=m_2x$ be the straight lines represented by the given equation, then

$m_1+m_2=-\dfrac{2h}{b}$ and $m_1m_2=\dfrac{a}{b}$. The perpendicular pair is

$$(m_1y+x)(m_2y+x)=0,$$

or
$$m_1m_2y^2+xy(m_1+m_2)+x^2=0.$$

Substituting for m_1m_2 and m_1+m_2, and simplifying, we get

$$ay^2-2hxy+bx^2=0.$$

5. Find the condition that one of the two lines

$$ax^2+2hxy+by^2=0$$

may be perpendicular to one of the lines given by

$$a'x^2+2h'xy+b'y^2=0.$$ *(Ranchi, 1968)*

Ans. $4(ah'+hb')(ha'+bh')+(aa'-bb')^2=0.$

6. Show that the product of the perpendiculars from the point (x', y') on the lines

$$ax^2+2hxy+by^2=0$$

is equal to

$$\frac{ax'^2+2hx'y'+by'^2}{\sqrt{(a-b)^2+4h^2}}.$$

(U. P. P. C. S., 1968; Lucknow, 1963)

7. If the lines $ax^2+2hxy+by^2=0$ be two sides of a parallelogram and the line $lx+my=1$ be one of its diagonals, show that the equation to the other diagonal is

$$y(bl-hm)=x(am-hl).$$

8. If p_1, p_2 be the perpendiculars from (x, y) on the straight lines $ax^2+2hxy+by^2=0$, prove that

$$(p_1^2+p_2^2)[(a-b)^2+4h^2]=2(a-b)(ax^2-by^2)$$
$$+4h(a+b)xy+4h^2(x^2+y^2).$$

9. Show that if one of the straight lines given by the equation

$$ax^2+2hxy+by^2=0$$

coincides with one of those given by

$$a'x^2+2h'xy+b'y^2=0,$$

then

$$(ab'-a'b)^2+4(ah'-ha')(bh'-b'h)=0.$$

(Allahabad, 1964; Bihar, 1965)

Hint. If $y=mx$ be the common straight line, then $a+2hm+bm^2=0$ and $a'+2h'm+b'm^2=0$. Eliminate m.

5.3 Equation to the bisector of angles between the lines $ax^2+2hxy+by^2=0$.

Let the two straight lines represented by the given equation be $y=m_1x$ and $y=m_2x$. As in § 3.7, the equations to the bisectors are

$$\frac{y-m_1x}{\sqrt{1+m_1^2}}=\pm\frac{y-m_2x}{\sqrt{1+m_2^2}}.$$

The two bisectors can be expressed in one equation which is

$$\left(\frac{y-m_1x}{\sqrt{1+m_1^2}}+\frac{y-m_2x}{\sqrt{1+m_2^2}}\right)\left(\frac{y-m_1x}{\sqrt{1+m_1^2}}-\frac{y-m_2x}{\sqrt{1+m_2^2}}\right)=0,$$

or
$$(1+m_2^2)(y-m_1x)^2-(1+m_1^2)(y-m_2x)^2=0,$$

or
$$x^2(m_1^2-m_2^2)-2xy(m_1-m_2)(1-m_1m_2)+y^2(m_2^2-m_1^2)=0$$

or
$$x^2-y^2=2xy\frac{1-m_1m_2}{m_1+m_2}.$$

Substituting $\dfrac{a}{b}$ for m_1m_2 and $-\dfrac{2h}{b}$ for m_1+m_2 as in § 5.12, the above equation becomes

$$x^2-y^2=2xy\frac{a-b}{2h}.$$

that is

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}.$$

Examples

1. Prove that the straight lines

$$ax^2+2hxy+by^2+\lambda(x^2+y^2)=0$$

have the same pair of bisectors for all values of λ. Interpret the case $\lambda=-(a+b)$.

2. Show that the bisectors of the angles between the lines

$$(ax+by)^2=3(bx-ay)^2$$

are respectively parallel and perpendicular to the line

$$ax+by+c=0.$$

3. Show that the pair of lines $ax^2+2hxy+by^2=0$ is equally inclined to the pair of lines

$$a^2x^2+2h(a+b)xy+b^2y^2=0. \qquad \textit{(Punjab, 1964)}$$

Hint. The two pairs have the same set of bisectors.

4. Show that the angle between one of the lines given by $ax^2+2hxy+by^2=0$ and one of the lines given by $ax^2+2hxy+by^2+\lambda(x^2+y^2)=0$ is equal to the angle between the other two lines of the system.

5. Show that the line $Ax+By+C=0$ and the two lines

$$(Ax+By)^2-3(Ay-Bx)^2=0$$

form an equilateral triangle.

Hint. Use the result of Ex. 2.

6. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines intersecting in (α, β), show that the equation to the line pair bisecting the angles between them is

$$\frac{(x-\alpha)^2-(y-\beta)^2}{a-b}=\frac{(x-\alpha)(y-\beta)}{h}.$$

5.4 Condition that the general equation of the second degree should represent a pair of straight lines.

Let the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad ...(1)$$

represents a pair of straight lines and let (x_1, y_1) be the point of intersection of the lines.

If the origin is transferred to (x_1, y_1) the axes remaining parallel to their original directions, equation (1) transforms into

$$a(X+x_1)^2+2h(X+x_1)(Y+y_1)+b(Y+y_1)^2$$
$$+2g(X+x_1)+2f(Y+y_1)+c=0,$$

or
$$aX^2+2hXY+bY^2+2(ax_1+hy_1+g)X+2(hx_1+by_1+f)Y$$
$$+ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0. \qquad ...(2)$$

Since equation (2) represents a pair of straight lines passing through the origin for the new set of axes, it must be a homogeneous equation of the second degree in X and Y.

Hence the coefficients of X and Y and the constant term in (2) must separately vanish.

Thus

$$ax_1+hy_1+g=0, \qquad ...(3)$$
$$hx_1+by_1+f=0, \qquad ...(4)$$

and
$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0. \qquad ...(5)$$

Multiplying (3) by x_1, (4) by y_1, adding and subtracting the result from (5), we get

$$gx_1+fy_1+c=0. \qquad ...(6)$$

Eliminating x_1, y_1 from (3), (4) and (6), we obtain

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0,$$

which is the condition that equation (1) should represent a pair of straight lines.

Expanding the above determinant, the condition that the general equation of the second degree should represent two straight lines is

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

Corollary. The coordinates of the point of intersection of the straight lines represented by the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

are obtained by solving for x_1 and y_1 the equations

$$ax_1+hy_1+g=0,$$
$$hx_1+by_1+f=0.$$

Hence,

$$x_1=\frac{hf-bg}{ab-h^2}, \quad y_1=\frac{gh-af}{ab-h^2}.$$

If $ab-h^2=0$, the point of intersection shifts to infinity and the lines become parallel.

The above method fails when the point of intersection of the lines lies at infinity. We may then proceed as follows :

Aliter.

The equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a pair of straight lines if the expression on the left can be broken up into two linear factors of the type

$$lx+my+n \text{ and } l'x+m'y+n'.$$

We then have

$$ax^2+2hxy+by^2+2gx+2fy+c\equiv(lx+my+n)\,(l'x+m'y+n'),$$

which gives

$$ll'=a,\ mm'=b,\ nn'=c,\ lm'+ml'=2h,$$
$$ln'+nl'=2g,\ mn'+nm'=2f.$$

Multiplying the last three results together, we obtain

$$2ll'mm'nn'+ll'\,(m^2n'^2+n^2m'^2)+mm'\,(n^2l'^2+l^2n'^2)$$
$$+nn'\,(l^2m'^2+m^2l'^2)=8fgh,$$

which reduces to

$$2abc+a\,(4f^2-2bc)+b\,(4g^2-2ac)+c\,(4h^2-2ab)=8fgh,$$

or $$abc+2fgh-af^2-bg^2-ch^2=0.$$

The student should see that this is a general method and applies equally when the lines are not parallel.

Example. *Prove that the general equation*

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents two parallel straight lines if

$$h^2=ab \text{ and } bg^2=af^2. \qquad (U.\ P.\ P.\ C.\ S.,\ 1977)$$

Prove also that the distance between them is

$$\frac{2\sqrt{(g^2-ac)}}{\sqrt{[a\,(a+b)]}}. \qquad (Ranchi,\ 1968)$$

Solution. Substituting $h^2=ab$ in the general equation, we get

$$ax^2+2\sqrt{(ab)}\ xy+by^2+2gx+2fy+c=0,$$

or $$(x\sqrt{a}+y\sqrt{b})^2+2\,(gx+fy)+c=0,$$

which will represent two straight lines, obviously,

if $$gx+fy\equiv\lambda\,(x\sqrt{a}+y\sqrt{b})$$

i.e., if $$g=\lambda\sqrt{a},\ f=\lambda\sqrt{b}$$

i.e., if $$\lambda=\frac{g}{\sqrt{a}}=\frac{f}{\sqrt{b}}$$

i.e., if $$bg^2=af^2,$$

and the equation to the straight lines becomes

$$(x\sqrt{a}+y\sqrt{b})^2+\frac{2g}{\sqrt{a}}\,(x\sqrt{a}+y\sqrt{b})+c=0.$$

The two lines are therefore

$$x\sqrt{a}+y\sqrt{b}=\frac{-g\pm\sqrt{g^2-ac}}{\sqrt{a}}$$

or $$x\sqrt{a}+y\sqrt{b}+\frac{g+\sqrt{g^2-ac}}{\sqrt{a}}=0 \qquad \ldots(1)$$

and $$x\sqrt{a}+y\sqrt{b}+\frac{g-\sqrt{g^2-ac}}{\sqrt{a}}=0. \qquad \ldots(2)$$

The lines are obviously parallel. To determine the distance between them take any point (x, y) on one of them, say on (2). The perpendicular from this point on (1) is

$$\frac{\sqrt{a}\,(x\sqrt{a}+y\sqrt{b})+g+\sqrt{g^2-ac}}{\sqrt{a\,(a+b)}}$$

Since (x, y) lies on (2),

$$\sqrt{a}\,(x\sqrt{a}+y\sqrt{b})+g=\sqrt{(g^2-ac)}.$$

Hence the distance between the lines is

$$\frac{2\sqrt{(g^2-ac)}}{\sqrt{[a\,(a+b)]}}.$$

5.5 Sufficiency of the condition.

We have seen that if the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines, then

$$\Delta\equiv\begin{vmatrix} a & g & h \\ h & b & f \\ g & f & c \end{vmatrix}=0.$$

This shows $\Delta=0$ is the *necessary* condition that the general equation of the second degree should represent two straight lines. We shall now show that this condition is *sufficient* also.

If $\Delta=0$, we can find x_1 and y_1 to satisfy the three linearly independent equations

$$ax_1+hy_1+g=0,$$
$$hx_1+by_1+f=0,$$
$$gx_1+fy_1+c=0.$$

If the origin is now transferred to (x_1, y_1) as determined by any two of the above equations, the general equation of the second degree is seen reduced to

$$aX^2+2hXY+bY^2=0$$

which is the equation to a pair of straight lines.

5.6 Angle between the lines represented by the equations

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

If the origin is transferred to the point of intersection of the lines, the equation reduces to

$$aX^2+2hXY+bY^2=0.$$

The lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ are, therefore, parallel to the pair of lines

$$ax^2+2hxy+by^2=0$$

through the origin.

The angle between the given lines is, therefore,

$$\tan^{-1}\left(\frac{2\sqrt{(h^2-ab)}}{a+b}\right).$$

The lines are parallel if $h^2=ab$, and perpendicular if $a+b=0$.

Examples

1. Show that the equation

$$3x^2+7xy+2y^2+5x+5y+2=0$$

represents a pair of straight lines.

2. Show that the equation

$$2x^2-5xy+2y^2-3x+3y+1=0$$

represents two straight lines intersecting at an angle $\tan^{-1}\frac{3}{4}$.

3. For what values of h does the equation

$$3x^2+2hxy-3y^2-40x+30y-75=0$$

represents two straight lines ?

Ans. $h=4$.

4. Find the equation of the two straight lines passing through (1, 1) and parallel to the straight lines

$$2x^2+5xy+3y^2+2x-1=0. \qquad \text{(Lucknow, 1962)}$$

Ans. $2x^2+6xy+3y^2-10x-12y+11=0$.

5. Find λ in order that the equation

$$x^2+\lambda xy+y^2-5x-7y+6=0$$

may represent two straight lines. Write down the equation to the lines parallel to these and passing through the origin.

(*Roorkee*, 1967)

Ans. $\frac{5}{2}, \frac{10}{3}$; $2x^2+5xy+2y^2=0$.

$3x^2+10xy+3y^2=0$.

6. Show that the four lines given by the equations

$$3x^2+8xy-3y^2=0$$

and $\quad 3x^2+8xy-3y^2+2x-4y-1=0$

form a square.

Find the equations to the diagonals of the square.

Ans. $2x=4y+1$, $2x+y=0$.

5.7 Lines joining the origin to the intersections of a curve and a line.

Let us find the equations to the pair of straight lines joining the points where the curve

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad ...(1)$$

is intersected by the straight line

$$lx+my=1. \qquad ...(2)$$

Now, we know that the equation to a pair of straight lines passing through the origin is a homogeneous equation of the second degree. If this pair passes through the points of intersection of (1) and (2), the homogeneous equation will be obtained with the help of (1) and (2).

Using (2) to make (1) homogeneous, we get

$$ax^2+2hxy+by^2+2\,(gx+fy)\,(lx+my)+c\,(lx+my)^2=0. \qquad ...(3)$$

The values of x and y which satisfy (1) and (2) simultaneously also satisfy (3). Equation (3), therefore, represents a pair of straight lines passing through the points of intersection of (1) and (2) and the origin.

Corollary. The equation to the n straight lines joining the origin to the points of intersection of a curve of the n^{th} degree and a straight line is obtained by making the equation of the curve homogeneous of the n^{th} degree with the help of the equation of the straight line.

Examples

1. Show that the straight lines joining the origin to the points of intersection of the line $7x-y+2=0$ and the curve $2x^2+y^2+x+y=0$ are at right angles to one another.

2. Prove that the angle between the straight lines joining the origin to the intersections of the straight line $y=3x+2$ with the curve

$$x^2+2xy+3y^2+4x+8y-11=0 \text{ is } \tan^{-1}\left(\frac{2\sqrt{2}}{3}\right).$$

3. Prove that the straight lines joining the origin to the points of intersection of the straight line $kx+hy=2hk$ and the curve $(x-h)^2+(y-k)^2=c^2$ are at right angles if $h^2+k^2=c^2$. (*Lucknow*, 1964)

4. Show that all chords of the curve

$$3x^2-y^2-2x+4y=0$$

which subtend a right angle at the origin pass through a fixed point.

Solution. Let $lx+my=1$ be one such chord. Equation to the pair of straight lines joining the origin to the intersection of the curve and this chord is

$$3x^2-y^2-2(x-2y)(lx+my)=0.$$

This subtends a right angle at the origin. Therefore,

$$3-2l-1+4m=0, \text{ or } l=1+2m.$$

Substituting for l in the equation to the chord, we obtain

$$x-1+m(2x+y)=0,$$

which shows that it passes through the fixed point $(1, -2)$.

5. Show that the straight lines joining the origin to the points of intersection of the two curves

$$ax^2+2hxy+by^2+2gx=0,$$

and

$$a'x^2+2h'xy+b'y^2+2g'x=0$$

will be at right angles to one another, if

$$g(a+b)=g(a'+b').$$ (*U. P. C. S.*, 1976)

Hint. Eliminate x from the given equations.

Examples on Chapter V

1. Show that the following equations represent pairs of straight lines :

(*i*) $x^2+xy-2y^2+x+5y-2=0$.

(*ii*) $2x^2-xy-y^2+5x+y+2=0$.

(*iii*) $4x^2+12xy+9y^2+10x+15y+4=0$.

(*iv*) $3x^2+10xy+3y^2-16x-16y+16=0$.

(*v*) $6x^2+7xy+20y^2+x+14y-2=0$.

(*vi*) $3x^2+10xy+8y^2+14x+22y+15=0$.

(*vii*) $25x^2+90xy+81y^2-40x-72y+7=0$.

2. If $6x^2-11xy-10y^2+19y+c=0$ represents two straight lines, find the equations of the lines and the tangent of the angle between them. (*Roorkee*, 1962)

Ans. $6x^2-11xy-10y^2+19y-6=0$, $\tan^{-1}\dfrac{19}{4}$.

3. If the pairs of straight lines

$x^2-2pxy-y^2=0$ and $x^2-2qxy-y^2=0$ be such that each pair bisects the angles between the other pair, prove that $pq=-1$.

(*Rajasthan*, 1963; *Delhi*, 1970)

4. Prove that the equation

$$3x^2-8xy-3y^2-29x+3y-18=0$$

represents two straight lines. Find their point of intersection and the angle between them.

Ans. $(-3/2, -5/2)$; $90°$.

5. Find the equation of the pair of straight lines obtained by joining the origin to the points of intersection of the straight line $y=mx+c$ and the circle $x^2+y^2=a^2$, and prove that these are at right angles if $2c^2=a^2(1+m^2)$. (*Vikram*, 1966)

6. Show that the equation $x^2-xy-6y^2-3x+14y-4=0$ represents a pair of straight lines inclined at $45°$ to each other; find also their point of intersection.

Ans. (2, 1).

7. If a and b are positive numbers, for what range of values of k can a real λ be found such that the equation

$$ax^2+2\lambda xy+by^2+2k(x+y+1)=0$$

represents a pair of straight lines.

Ans. k^2 does not lie between a and b.

8. Show that the straight lines

$$(a^2-3b^2)x^2+8abxy+(b^2-3a^2)y^2=0$$

form with the line $ax+by+c=0$ an equilateral triangle of area

$$\frac{c^2}{(a^2+b^2)\sqrt{3}}.$$ (*Agra*, 1965)

Hint. See Q. 5 § 5.3.

9. Show that the lines joining the origin to the points common to $3x^2+5xy-3y^2+2x+3y=0$ and $3x-2y=1$ are at right angles.

10. If the diagonals of the parallelogram formed by

$$px+qy+r=0 \qquad p'x+q'y+r=0$$
$$px+qy+r'=0 \qquad p'x+q'y+r'=0$$

are at right angles, prove that $p^2+q^2=p'^2+q'^2$.

Hint. The distance between each parallel lines is equal.

11. Show that $y^2-4y+3=0$ and $x^2+4xy+4y^2+5x+10y+4=0$ represent lines forming a parallelogram, and find the lengths of the sides.

Ans. $(2\sqrt{5}, 3)$.

12. A parallelogram is formed by the lines $ax^2+2hxy+by^2=0$ and the lines through (p, q) parallel to them. Prove that the equation of the diagonal which does not pass through the origin is

$$(2x-p)(ap+hq)+(2y-q)(hp+bq)=0.$$

Also show that the area of the parallelogram is

$$\frac{ap^2+2hpq+bq^2}{2\sqrt{h^2-ab}}.$$

(*Lucknow*, 1962; *Andhra*, 1963)

Hint. If $S_1=0$ represents the given line pair and

$$S_2\equiv a(x-p)^2+2h(x-p)(y-q)+b(y-q)^2=0$$

the other parallel pair of lines, the equation of the required diagonal is

$$S_1-S_2=0.$$

13. Show that the area of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$

is

$$\frac{n^2\sqrt{h^2-ab}}{am^2-2hlm-bl^2}.$$

14. A pair of perpendicular straight lines is drawn through the origin forming with the line $2x+3y=6$ an isosceles triangle right angled at the origin. Find the equation of the line pair and the area of the triangle. *Ans.* $5x^2-24xy-5y^2=0$, $(36/13)$.

15. A point moves so that the distance between the feet of the perpendiculars from it on the lines $ax^2+2hxy+by^2=0$ is a constant $2K$. Show that its equation is

$$(x^2+y^2)(h^2-ab)=K^2[(a-b)^2+4h^2].$$ (*Lucknow*, 1958)

16. The orthocentre of the triangle formed by the lines

$$ax^2+2hxy+by^2=0 \text{ and } lx+my=1$$

is (α, β). Show that

$$\frac{\alpha}{l}=\frac{\beta}{m}=\frac{a+b}{am^2+2hlm+bl^2}$$ (*Agra*, 1962)

17. A triangle has the line $ax^2+2hxy+by^2=0$ for two of its sides and the point (p, q) for the orthocentre. Prove that the equation to the third side is

$$(a+b)(px+qy)=aq^2-2hpq+bp^2.$$ (*Lucknow*, 1960)

18. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, show that the square of the distance of their point of intersection from the origin is

$$\frac{c(a+b)-g^2-f^2}{ab-h^2}.$$

19. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, prove that the product of the perpendiculars from the origin on these lines is

$$\frac{c}{\sqrt{[(a-b)^2+4h^2]}}.$$

20. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, equidistant from the origin, prove that

$$f^4-g^4=c(bf^2-ag^2).$$

21. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, prove that the area of the triangle formed by the lines and the axis of x is

$$\frac{g^2-ac}{\sqrt{h^2-ab}}.$$ (*Rajasthan*, 1962)

22. The equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines at right angles to one another, prove that the square of the distance of their point of intersection from the origin is

$$\frac{f^2+g^2}{h^2+b^2}.$$ (*U. P. C. S.*, 1975)

23. If $u\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, prove that the equation of the third pair of straight lines through the points where these lines meet the coordinate axes is

$$cu+4(fg-ch)xy=0.$$

(*Andhra*, 1962; *I. A. S.*, 1966)

Hint. Determine λ such that $u+\lambda xy=0$ may represent two straight lines.

24. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, prove that the area of the triangle formed by their bisectors and the axis of x is

$$\frac{\sqrt{(a-b)^2+4h^2}}{2h}\cdot\frac{ca-g^2}{ab-h^2}.$$ (*I. A. S.*, 1976)

25. If $ax^2+2hxy+by^2+2gx+2fy+c=0$

and $ax^2+2hxy+by^2-2gx-2fy+c=0$,

each represents a pair of straight lines, prove that the area of the parallelogram enclosed by them is

$$\frac{2c}{\sqrt{h^2-ab}}.$$ (*Rajasthan*, 1961)

26. If two of the lines given by

$$ax^3+3bx^2y+3cxy^2+dy^3=0$$

are at right angles, show that

$$a^2+3ac+3bd+d^2=0.$$

(*Agra*, 1966; *U. P. C. S.*, 1969)

Hint. All the lines pass through the origin. If m be the slope of any one line, then $a+3bm+3cm^2+dm^3=0$. The condition is obtained by putting the product of the two roots equal to -1.

27. Prove that the two of the lines represented by the equation

$$ax^4+bx^3y+cx^2y^2+dxy^3+ay^4=0$$

will bisect the angle between the other two, if $c+6a=0$ and $b+d=0$.

28. Show that the equation

$$a(x^4+y^4)-4bxy(x^2-y^2)+6cx^2y^2=0.$$

represents two pairs of straight lines at right angles and if $2b^2=a^2+3ac$, the two pairs will coincide. (*I. A. S.*, 1977)

Hint. Let the two pairs of straight lines be

$$x^2-\lambda xy-y^2=0 \text{ and } ax^2-\mu xy-ay^2=0.$$

Write the combined equation and compare it with the given equation. The two pairs will coincide if $\mu=a\lambda$.

29 A variable line through (α, β) meets the lines $ax^2+2hxy+by^2=0$ in P and Q. Show that the locus of the middle point of PQ is

$$ax^2+2hxy+by^2=\alpha(ax+hy)+\beta(hx+by).$$

30. The base of a triangle passes through a fixed point (f, g) and its sides are respectively bisected at right angles by the pair of line $ax^2+2hxy+by^2=0$. Prove that the locus of the vertex of the triangle is

$$(a+b)(x^2+y^2)+2h(gx+fy)+(a-b)(fx-gy)=0.$$

CHAPTER VI +

THE GENERAL EQUATION OF THE SECOND DEGREE

6.1 The Conic Section. If a cone having a circular base (not necessarily right) is cut by a plane, the section is one of the five curves, *viz.*, a pair of straight lines, a circle, a parabola, an ellipse or a hyperbola. The shape of the section depends upon the position of the cutting plane; for example, the section by a plane through the vertex will be a pair of straight lines and the section by a plane parallel to the base will be a circle. If the plane neither passes through the vertex nor is parallel to the base, the section will be one of the curves—parabola, ellipse or hyperbola.

The above five curves are for this reason called the conic sections. They all share what is called the **focus-directrix** property. That is, each one of them is the locus of a point which moves such that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line. The fixed point is called the **focus**, the fixed line the **directrix**, and the constant ratio the **eccentricity** which is denoted by the letter 'e'. The cone* is an ellipse, parabola or hyperbola according as $e<1$, $=1$ or >1. The pair of straight lines and the circle are limiting case of the parabola and the ellipse respectively. The parabola too is a limiting case of the ellipse.

6.2 Equation to a Conic Section. We shall show that every conic section is represented by an equation of the second degree.

We have seen in the preceding chapter that a pair of straight lines is represented by an equation of the form $ax^2+2hxy+by^2+2gx+2fy+c=0$.

We shall see in the next chapter that the same equation represents a circle under a different set of conditions.

If the section be an ellipse, or parabola, or hyperbola let e be the eccentricity, (h, k) the coordinates of the focus and $Ax+By+C=0$ the equation of the directrix referred to rectangular axes in the plane of the section. The distances of any point (x, y) on the curve under consideration from the locus and the directrix are respectively :

* Conic section is briefly written as conic.

$$\sqrt{(x-h)^2+(y-k)^2} \text{ and } \frac{Ax+By+C}{\sqrt{A^2+B^2}}.$$

Hence

$$(x-h)^2+(y-k)^2=e^2\frac{(Ax+By+C)^2}{A^2+B^2} \qquad ...(1)$$

which is an equation of the second degree of the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad ...(2)$$

and which represents the equation to a conic section.

If the axes were inclined at angle ω, instead of (1) we would have

$$(x-h)^2+(y-k)^2+2\,(x-h)\,(y-k)\cos\omega=e^2\frac{(Ax+By+C)^2\sin^2\omega}{(A^2+B^2-2AB\cos\omega)},$$

which again is an equation of the second degree of the same form as (2).

We shall prove the converse proposition, namely, that an equation of the second degree always represents a conic section in a later chapter, after the standard equations to the circle, parabola, ellipse and the hyperbola have been discussed. We may state here for the benefit of the student that the standard equation to the *circle* is $x^2+y^2=a$, the standard equation to the *parabola* is $y^2=4ax$, the standard equation to the *ellipse* is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and the standard equation to the *hyperbola* is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

where a and b have assigned values.

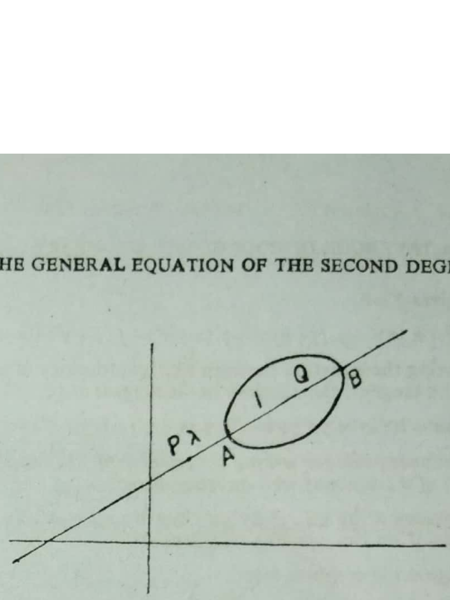
It should not be difficult to see that a suitable transformation of coordinate axes will reduce the general equation of the second degree to one or other of standard forms.

We now proceed to obtain equations to certain loci which are true for all conics and as easy to deduce for the general equation as for the standard ones.

6.3 Intersection of a line and a conic. Let a straight line drawn through the point $P\,(x', y')$, meet the conic

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

in A and let Q be any other point (x, y) on the line. Let A divides PQ in the ratio $\lambda : 1$.



The coordinates of A are $\left(\frac{\lambda x+x'}{1+\lambda}, \frac{\lambda y+y'}{1+\lambda}\right)$. Since A lies on the conic,

$$a\left(\frac{\lambda x+x'}{1+\lambda}\right)^2+2h\left(\frac{\lambda x+x'}{1+\lambda}\right)\left(\frac{\lambda y+y'}{1+\lambda}\right)+b\left(\frac{\lambda y+y'}{1+\lambda}\right)^2$$
$$+2g\frac{\lambda x+x'}{1+\lambda}+2f\frac{\lambda y+y'}{1+\lambda}+c=0,$$

or

$$\boxed{\lambda^2S+2\lambda T+S'=0.} \qquad ...(1)$$

where $T\equiv axx'+h\,(xy'+yx')+byy'+g\,(x+x')+f\,(y+y')+c$,

and $S'\equiv ax'^2+2hx'y'+by'^2+2gx'+2fy'+c$.

Equation (1) is of second degree in λ and accordingly gives two values of λ which may be real and distinct, real and coincident, or imaginary.

Every straight line therefore cuts a conic at two points.

Note. Imaginary points are as important in analytical geometry as the imaginary numbers are in other branches of mathematics. To the students of pure geometry a line in the plane of a conic either meets it or does not meet it. In analytical geometry, we say that any line in the plane of a conic meets it in two points. This is so because we can have imaginary points, *i.e.*, points whose coordinates are of the form $p+iq$, p, q being real and $i=\sqrt{(-1)}$. Points with imaginary coordinates cannot be shown in a figure but their algebraical significance is a well established fact.

6.31 Tangent at (x', y'). We shall now obtain the equation to the tangent at the point (x', y') of the conic $S=0$.

Using the equation (1) of § 6.3 the equation in λ is

$$\lambda^2S+2\lambda T+S'=0.$$

Since (x', y') lies on the conic $S=0$, we have therefore one value of $\lambda=0$. Further, since the two points of contact coincide at (x', y'), the other values of λ must also be equal to zero.

This gives $T=0$,

i.e., $axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c=0.$

This being the equation between the coordinates of any point (x', y') on the tangent, the *equation to the tangent at* (x', y') is

$$T\equiv axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c=0.$$

The following rule for writing the equation of the tangent at any point (x', y') of a conic may well be remembered.

(*i*) Replace x^2 by xx', y^2 by yy', $2xy$ by $xy'+yx'$, $2x$ by $x+x'$ and $2y$ by $y+y'$, in the equation of the conic.

(*ii*) Retain the constant term.

6.32 Condition of tangency of the line $lx+my+n=0$.

If the given line touches the conic $S=0$ at (x', y') the equation

$$lx+my+n=0$$

must be the same as

$$axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c=0,$$

or $$x(ax'+hy'+g)+y(hx'+by'+f)+gx'+fy'+c=0.$$

Comparing coefficients, we have

$$\frac{ax'+hy'+g}{l}=\frac{hx'+by'+f}{m}=\frac{gx'+fy'+c}{n}=\lambda, \text{ say.}$$

Therefore

$$ax'+hy'+g-l\lambda=0,$$
$$hx'+by'+f-m\lambda=0,$$
$$gx'+fy'+c-n\lambda=0.$$

Since (x', y') lies on the given line, we also have

$$lx'+my'+n=0.$$

Eliminating x', y' and λ from the above four equations, we get

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix}=0,$$

which is the required *condition of tangency*.

When expanded, the above determinant gives $Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0$, where A, B, C, etc., are the cofactors of a, b, c, etc., in

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

The coordinates of the point of contact, if required, are obtained from the equations

$$\frac{ax'+hy'+g}{l}=\frac{hx'+by'+f}{m}=\frac{gx'+fy'+c}{n}.$$

Examples

Ex. 1. Find the equation of the tangent at the point $(-2, 1)$ to the conic

$$x^2+2xy-y^2+2x+4y+1=0.$$

Ans. $y=1$.

Ex. 2. Show that the line $4x+5y-14=0$ touches the conic $x^2+4xy+y^2-2x+2y-15=0$ at the point $(1, 2)$.

Ex. 3. Find the condition that the straight line $lx+my=1$ may touch the conic

$$(lx-my)^2-2(l^2+m^2)(lx+my)+(l^2+m^2)^2=0.$$

Ans. $l^2+m^2=2$.

6.33 Pair of tangents from a given point.

Let (x', y') be the given point, and, as before, let (x, y) be the coordinates of a point on the line drawn through (x', y') to meet the conic. If the join of (x', y') and (x, y) is divided in the ratio $\lambda : 1$ by the point of intersection with the conic, the quadratic in λ [equation (1), § 6.3] is

$$\lambda^2S+2\lambda T+S'=0.$$

If the line is a tangent to the conic, the two points of intersection coincide. Consequently the two values of λ are equal, the condition for which is

$$\boxed{SS'=T^2}$$

and this is the equation to the pair of tangents from (x', y') to $S=0$.

Examples

Ex. 1. *Find the angle between the pair of tangents drawn from (x', y') to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Hence find the locus of the vertices of equilateral triangles circumscribing the ellipse.*

Solution. The equation to the pair of tangents from (x', y') is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right)^2.$$

The coefficient of $x^2=\frac{1}{a^2}\left(\frac{y'^2}{b^2}-1\right)$.

The coefficient of $y^2=\frac{1}{b^2}\left(\frac{x'^2}{a^2}-1\right)$.

The coefficient of $2xy=-\frac{x'y'}{a^2b^2}$.

The angle between the tangents is therefore

$$\tan^{-1}\left\{\frac{2\sqrt{\frac{x'^2y'^2}{a^4b^4}-\frac{1}{a^2b^2}\left(\frac{y'^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}-1\right)}}{\frac{1}{a^2}\left(\frac{y'^2}{b^2}-1\right)+\frac{1}{b^2}\left(\frac{x'^2}{a^2}-1\right)}\right\}$$

i.e., $$\tan^{-1}\left\{\frac{2ab\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1}}{x'^2+y'^2-a^2-b^2}\right\}.$$

If the tangents form sides of an equilateral triangle the angle between them is 60°. This gives

$$\sqrt{3}\,(x'^2+y'^2-a^2-b^2)=2ab\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1}.$$

Hence the locus of (x', y') is

$$3\,(x^2+y^2-a^2-b^2)^2=4a^2b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).$$

Ex. 2. A pair of tangents to the conic $ax^2+by^2=1$ intercepts a constant distance $2c$ on the axis of y; prove that the locus of their points of intersection is the curve

$$ax^2\,(ax^2+by^2-1)=bc^2\,(ax^2-1)^2.$$

Ex. 3. Pair of tangents are drawn to the conic $ax^2+by^2=1$ so as to be always parallel to the pair of straight lines $a'x^2+2h'xy+b'y^2=0$. If a', h', b' vary such that $a'B-2h'H+b'A=0$, A, H, B being constants, then show that the locus of the point of intersection of these tangents is the conic

$$Ax^2+2Hxy+By^2=\frac{A}{a}+\frac{B}{b}.$$

Hint. Compare coefficients of the second degree terms in the equation of the pair of tangents with those in the equation $a'x^2+2h'xy+b'y^2=0$.

6.34 Chord of contact of tangents drawn from (x', y') to $S=0$.

Let the coordinates of the points of contact of tangents drawn from (x', y') to $S=0$ be (x_1, y_1) and (x_2, y_2).

The equation of the tangent at (x_1, y_1) is

$axx_1+h\,(xy_1+yx_1)+byy_1+g\,(x+x_1)+f\,(y+y_1)+c=0.$

This passes through (x', y'). Therefore,

$$ax'x_1+h\,(x'y_1+y'x_1)+by'y_1+g\,(x'+x_1)+f\,(y'+y_1)+c=0 \quad ...(1)$$

Similarly, since the tangent at (x_2, y_2) also passes through (x', y'), we have

$$ax'x_2+h\,(x'y_2+y'x_2)+by'y_2+g\,(x'+x_2)+f\,(y'+y_2)+c=0 \quad ...(2)$$

From (1) and (2) we easily see that (x_1, y_1) and (x_2, y_2) both lie on the straight line

$$\boxed{axx'+h\,(xy'+yx')+byy'+g\,(x+x')+f\,(y+y')+c=0,}$$

which is therefore the equation to the chord of contact of tangents that can be drawn from (x', y') to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$.

This is the same as the equation to the tangent when (x', y') lies on the conic and can be written as $T=0$.

Note. The conic and the point being given, the chord of contact is a real line. Since it always passes through the two points of contact of tangents that can be drawn from the given point to the conic, which may be imaginary, we may have here the case of a real straight line passing through two imaginary points.

Examples

Ex. 1. *Show that if tangents be drawn to the parabola $y^2=4ax$ from a point on the line $x+4a=0$, their chord of contact will subtend a right angle at the vertex (origin).*

Solution. The vertex of the parabola is the origin. A point on $x+4a=0$ is $(-4a, k)$. Chord of contact of tangents from this point is

$$ky=2a\,(x-4a),$$

or

$$2ax-ky=8a^2.$$

Making the equation of the parabola homogeneous with the help of the above equation we get the equation of the join of the vertex with the point of contact as

$$y^2-\frac{4ax\,(2ax-ky)}{8a^2}=0,$$

or

$$2ay^2-2ax^2+kxy=0.$$

The two lines represented by this equation are at right angles.

Ex. 2. Find the locus of points which are such that the chords of contact of tangents drawn from them to the conic $ax^2+by^2=1$ form a triangle of constant area with the coordinate axes.

(*Lucknow*, 1965)

Ans. $xy=$constant.

6.4. Pole and Polar. *Definition.* If any secant, PAB, through a given point P, meets a conic in A and B, then the locus of Q, the harmonic conjugate of P with respect to A and B, *viz.*, such that $\dfrac{1}{PA}+\dfrac{1}{PB}=\dfrac{2}{PQ}$, is defined as the *polar* of P with respect to the conic, and P is called its *pole*.

Referring to the figure of § 6.3 let P and Q be the points (x', y') and (α, β), and let the equation to PAB be (chapter III)

$$\frac{x-x'}{l}=\frac{y-y'}{m}=r,$$

r being the distance of (x, y) from (x', y').

Then, $x=x'+lr$, $y=y'+mr$.

If the point (x, y) lies on the conic

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0,$$

we have
$$a(x'+lr)^2+2h(x'+lr)(y'+mr)+b(y'+mr)^2$$
$$+2g(x'+lr)+2f(y'+mr)+c=0,$$

i.e.,
$$r^2(al^2+2hlm+bm^2)+2r\{l(ax'+hy'+g)$$
$$+m(hx'+by'+f)\}+S'=0. \qquad \ldots(1)$$

This is a quadratic in r and its roots r_1, r_2 are the measures of PA and PB.

If PQ be equal to R, we have, if Q lies on the polar of P,

$$\frac{1}{r_1}+\frac{1}{r_2}=\frac{2}{R},$$

i.e.,
$$R=\frac{2r_1r_2}{r_1+r_2}=-\frac{S'}{l(ax'+hy'+g)+m(hx'+by'+f)}.$$

From the equation to the line,

$$\alpha-x'=lR,\ \beta-y'=mR.$$

Therefore,

$$(ax'+hy'+g)(\alpha-x')+(hx'+by'+f)(\beta-y')+S'=0,$$

or

$$a\alpha x'+h(\alpha y'+\beta x')+b\beta y'+g(\alpha+x')-f(\beta+y')+c=0.$$

Replacing α, β by current coordinates, the locus of Q is the straight line

$$\boxed{axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c=0,}$$

which is the *polar* of (x', y') with respect to the conic $S=0$.

If P is on the conic, the polar of P is the tangent at P.

Note 1. It is easy to notice that the polar of a point and the chord of contact of tangents that can be drawn from the point to the conic are one and the same thing. The polar of a point is consequently sometimes defined as the chord of contact of tangents from the point. We shall however treat this only as a property of polar.

Note 2. The polar of a point is also defined as the locus of the points of intersection of tangents at the extermities of chords through that point. This definition gives the equation of the polar easily.

For, if (x', y') be the given point and (α, β) a point on the polar, (x', y') lies on the chord of contact of tangents from (α, β) to the conic. Thus

$$a\alpha x'+h(\alpha y'+\beta x')+b\beta y'+g(\alpha+x')+f(\beta+y')+c=0,$$

from which the locus of (α, β), as before, is

$$axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c=0.$$

Note 3. The method of this article can also be used to determine the equations of the tangent and pair of tangents. The student should do this as an exercise.

Examples

Ex. 1. *Find the pole of the line* $x+y+9=0$ *with respect to the conic* $x^2-2xy+y^2-3x+y-2=0$.

Solution. Let the pole be (x', y').

Then the polar is

$$xx'-(xy'+yx')+yy'-\tfrac{3}{2}(x+x')+\tfrac{1}{2}(y+y')-2=0,$$

or
$$x(2x'-2y'-3)+y(2y'-2x'+1)-3x'+y'-4=0.$$

This is the same as the given line. Comparing coefficients, we have

$$2x'-2y'-3=2y'-2x'+1=\frac{y'-3x'-4}{9}.$$

Solving, $x'=2$, $y'=1$.

Hence the pole is $(2, 1)$.

Ex. 2. Show that the pole of the line $2x-13y-2=0$ with respect to the conic $x^2+y^2-4xy-2x-20y-11=0$ is $(1, -1)$.

Ex. 3. Show that the locus of poles with respect to the parabola $y^2=4ax$ of tangents of the hyperbola $x^2-y^2=a^2$ is the ellipse $4x^2+y^2=4a^2$. *(Jodhpur, 1968)*

6.41 Conjugate Points. We shall show that *if the polar of P passes through Q, then the polar of Q will pass through P.*

The polar of $P\ (x_1, y_1)$ with respect to the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is

$$axx_1+h\ (xy_1+yx_1)+byy_1+g\ (x+x_1)+f\ (y+y_1)+c=0.$$

If this passes through $Q\ (x_2, y_2)$,

$$ax_2x_1+h\ (x_2y_1+x_1y_2)+by_1y_2+g\ (x_2+x_1)+f\ (y_2+y_1)+c=0,$$

which is also seen to be the condition that P should lie on the polar of Q.

Two points such that each lies on the polar of the other are called conjugate points, and the condition that (x_1, y_1), (x_2, y_2) be conjugate points is

$$ax_1x_2+h(x_1y_2+x_2y_1)+by_1y_2+g(x_1+x_2)+f\ (y_1+y_2)+c=0.$$

6.42 Conjugate Line. We shall prove that *if the pole of a line $u_1=0$ lies on another line $u_2=0$, then the pole of $u_2=0$ lies on $u_1=0$.*

Let $(x_1\ y_1)$, (x_2, y_2) be respectively the poles of $u_1=0$, $u_2=0$. The line $u_1=0$ is the same as

$$axx_1+h\ (xy_1+yx_1)+byy_1+g\ (x+x_1)+f\ (y+y_1)+c=0. \quad ...(1)$$

and $u_2=0$ is the same as

$$axx_2+h\ (xy_2+yx_2)+byy_2+g\ (x+x_2)+f\ (y+y_2)+c=0. \quad ...(2)$$

If, now, (x_1, y_1) lies on (2),

$$ax_1x_2+h\ (x_1y_2+y_1x_2)\ +by_1y_2+g\ (x_1+x_2)+f\ (y_1+y_2)+c=0,$$

which is also the condition that (x_2, y_2) should lie on (1).

Hence if the pole of one lies on another, the pole of the second line will lie on the first.

Two such lines are said to be *conjugate lines*.

6.43 The condition that two given lines should be conjugate with respect to a given conic.

Let $$lx+my+n=0, \quad ...(1)$$

and $$l'x+m'y+n'=0, \quad ...(2)$$

be conjugate lines for the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Pole

Let (x', y') be the pool of (1) with respect to the conic, then (1) is the same as

$$axx'+h\ (xy'+yx')+byy'+g\ (x+x')+f\ (y+y')+c=0,$$

i.e., as

$$x\ (ax'+hy'+g)+y\ (hx'+by'+f)+gx'+fy'+c=0.$$

Comparing coefficients, we obtain

$$\frac{ax'+hy'+g}{l}=\frac{hx'+by'+f}{m}=\frac{gx'+fy'+c}{n}=\lambda, \text{ say.}$$

Therefore,

$$ax'+hy'+g-l\lambda=0, \quad ...(3)$$
$$hx'+by'+f-m\lambda=0, \quad ...(4)$$
$$gx'+fy'+c-n\lambda=0. \quad ...(5)$$

Also, (x', y') lies on (2), since (1) and (2) are conjugate lines. Hence

$$l'x'+m'y'+n'=0 \quad ...(6)$$

Eliminating x', y' and λ from (3), (4), (5) and (6), we have

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l' & m' & n' & o \end{vmatrix}=0,$$

which is the condition that the lines $lx+my+n=0$ and $l'x+m'y+n'=0$ should be conjugate with respect to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$.

6.44 Self-Polar triangle. If the vertices A, B, C of a triangle are such that any pair of them are conjugate points, the triangle is said to be *self-polar* and *self-conjugate*. Any two sides of such a triangle will be conjugate lines. For since the polar of A passes through both B and C, BC is the polar of A, so that each side of the triangle is the polar of the opposite vertex. The rest is obvious as each vertex is the meeting point of the other two sides of the triangle.

Example. Show that $lx+my+n=0$ and $l'x+m'y+n'=0$ are conjugate of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ if $a^2ll'+b^2mm'=nn'$.

6.5 Chord with given middle point. Let (x', y') be the middle point of a chord of the conic

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0.$$

As in § 6.4 the line

$$\frac{x-x'}{l}=\frac{y-y'}{m}=r \quad ...(1)$$

cuts the conic in two points whose distances from (x', y') are the roots of the equation

$$r^2(al^2+2hlm+bm^2)+2r\{(ax'+hy'+g)l+(hx'+by'+f)m\}+S'=0.$$

Since (x', y') is the middle point of the chord, the root of the above quadratic are equal in magnitude and opposite in sign. Their sum consequently vanishes and we have

$$(ax'+hy'+g)\,l+(hx'+by'+f)\,m=0 \qquad \ldots(2)$$

Assuming that the coefficients of l and m in equation (2) do not vanish simultaneously,* we get on eliminating $l : m$ from (1) and (2)

$$(ax'+hy'+g)(x-x')+(hx'+by'+f)(y-y')=0. \qquad \ldots(3)$$

Equation (3) *is the equation of the chord whose middle point is* (x', y').

Equation (3) can also be written as

$$axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c$$
$$=ax'^2+2hx'y+by'^2+2gx'+2fy'+c,$$

which is the same as

$$T=S'.$$

Note. The above result can be obtained equally easily by considering the equation in λ of § 6.3. In this case, the roots are λ_1 and $\dfrac{-\lambda_1}{1+2\lambda_1}$, so that the sum of the roots is equal to twice their product with minus sign.

Example

Ex. 1. Find the equation to the chord of the conic $x^2+2y^2=1$ whose middle point is $(-1, 2)$.

Ans. $x-4y+9=0$.

Ex. 2. Find the middle point of the chord $x+3y-2=0$ of the conic $x^2+xy-y^2=1$.

Ans. $(-1, 1)$.

Ex. 3. Show that locus of middle points of chords of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ passing through a fixed point (h, k) is

$$b^2x(x-h)+a^2y(y-k)=0.$$

Ex. 4 Chords of the parabola $y^2=4ax$ subtend a right angle at the vertex (origin). Find the locus of their middle points.

(*Punjab*, 1965)

Ans. $y^2-2ax+8a^2=0$.

*If they vanish simultaneously, then every chord passing through (x', y') is bisected there.

6.6 Centre. *Definition. The centre of a conic is a point such that all chords of the conic passing through it are bisected there.*

We shall obtain the centre of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

If (x', y') be the middle point of the chord

$$\frac{x-x'}{l}=\frac{y-y'}{m},$$

then, as in the preceding article,

$$(ax'+hy'+g)\,l+(hx'+by'+f)\,m=0.$$

If (x', y') is the centre of the conic, the above equation is true for all values of the ratio $l : m$. We then have

$$ax'+hy'+g=0 \qquad \ldots(1)$$

and

$$hx'+by'+f=0. \qquad \ldots(2)$$

The centre of the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ is thus obtained by solving (1) and (2) as simultaneous equations.

The coordinates of the centre are

$$\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right),$$

which can be written as

$$\left(\frac{G}{C}, \frac{F}{C}\right)$$

where C, F and G are respectively the cofactors of c, f and g in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Note. If $ab-h^2=0$, the centre lies at infinity. We shall see in a later chapter that under this condition the conic is, in general, a parabola.

Aliter. The centre (x', y') of the general conic can also be found as follows :

Transferring the origin to (x', y') the new axes remaining parallel to the old, the equation to the conic becomes (§ 5.4)

$$aX^2+2hXY+bY^2+2(ax'+hy'+g)X+2(hx'+by'+f)Y+S'=0.$$

Since the points (X, Y), $(-X, -Y)$ now lie on the conic,

$$(ax'+hy'+g)X+(hx'+by'+f)Y=0.$$

Scanned by CamScanner

And as the relation is true for all values of X and Y, we have, as before, the equations

$$ax'+hy'+g=0,$$

and

$$hx'+by'+f=0,$$

which give the coordinates of the centre.

The student familiar with Calculus, will notice that equations giving the centre are the ones obtained by *differentiating the equation to the conic partially with respect to x and partially with respect to y.*

6.7 The equation to the conic

$$\mathbf{ax^2+2hxy+by^2+2gx+2fy+c=0}$$

when the origin is transferred to the centre.

The coordinates (x', y') of the centre are given by

$$ax'+hy'+g=0 \qquad \ldots(1)$$

and

$$hx'+by'+f=0. \qquad \ldots(2)$$

On transferring the origin to (x', y'), the equation of the conic becomes*

$$aX^2+2hXY+bY^2+c'=0,$$

where $c'=ax'^2+2hx'y'+b'y'^2+2gx'+2fy'+c.$ ...(3)

Multiplying (1) by x', (2) by y', adding and using (3), we get

$$gx'+fy'+c-c'=0. \qquad \ldots(4)$$

Eliminating x' and y' between (1), (2) and (4),

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c-c' \end{vmatrix}=0,$$

$c' = gx' + fy' + c$
used in
unit IV

or

$$\Delta+\begin{vmatrix} a & h & 0 \\ h & b & 0 \\ g & f & -c' \end{vmatrix}=0,$$

or

$$\Delta-c'(ab-h^2)=0,$$

i.e.,

$$c'=\frac{\Delta}{ab-h^2},$$

where

$$\Delta\equiv\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

* See Aliter of § 6.6.

The value of c' from (4) is $\mathbf{gx'+fy'+c}$, which is used when the coordinates of the centre are known.

Corollary. The equation $ax^2+2hxy-by^2=1$ represents a conic whose centre is at the origin.

Examples

Ex. 1. Find the centre of the conic $x^2-xy-y^2+6x+3y+5=0$ and determine its equation referred to parallel axes through the centre.

Ans. $(3, 0)$; $x^2-xy-y^2=4$.

Ex. 2. Show that the equation of the conic

$$x^2-5xy+y^2+8x-20y+15=0$$

referred to parallel axes through the centre is

$$x^2-5xy+y^2=1.$$

6.8 Locus of the middle points of a system of parallel chords.

Let chords of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

be drawn parallel to the line $y=mx$.

If (x', y') be the middle point of one chord of the system, its equation is

$$(ax'+hy'+g)(x-x')+(hx'+by'+f)(y-y')=0.$$

The slope of the chord must be equal to m.

Hence,

$$-\frac{ax'+hy'+g}{hx'+by'+f}=m.$$

The required locus is therefore the straight line

$$(ax+hy+g)+m(hx+by+f)=0.$$

This passes through the point of intersection of the lines $ax+hy+g=0$ and $hx+by+f=0$, which is the centre of the given conic.

6.8.1 Conjugate Diameters.

Definition. *The locus of the middle points of a system of parallel chords of a conic is called a diameter. Two diameters are said to be conjugate when each bisects chords of the conic parallel to the other.*

We have seen in the preceding article that a diameter of a central conic passes through its centre. Let the diameter drawn parallel to $y=m'x$ bisects all chords of the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$, which are parallel to $y=mx$.

The equation to the diameter from the preceding article is

$$(ax+hy+g)+m\,(hx+by+f)=0,$$

or

$$x\,(a+hm)+y\,(h+bm)+g+fm=0.$$

Since by hypothesis this is parallel to $y=m'x$, we have

$$-\frac{a+hm}{h+bm}=m',$$

i.e.,

$$a+hm+(h+bm)\,m'=0,$$

i.e.,

$$a+h\,(m+m')+bmm'=0.$$

From the symmetry of this relation, in m and m', we see that the same is the condition in order that the diameter parallel to $y=mx$ may bisect chords parallel to $y=m'x$.

Hence *the diameters* $y=mx$ and $y=m'x$ *are conjugate if*

$$\mathbf{a+h\,(m+m')+bmm'=0.}$$

Examples

Ex. 1. *Find the condition that the lines*

$$Ax^2+2Hxy+By^2=0$$

may be conjugate diameters of the conic

$$ax^2+2hxy+by^2=1.$$ (*Lucknow*, 1964)

Solution. Let $y=mx$ and $y=m'x$ be conjugate diameters of the given conic.

Then $(y-mx)\,(y-m'x)=0$ must be identical with

$$Ax^2+2Hxy+By^2=0,$$

which gives

$$m+m'=-\frac{2H}{B} \text{ and } mm'=\frac{A}{B}.$$

But

$$a+h\,(m+m')+bmm'=0$$

Hence

$$a-h\,\frac{2H}{B}+b\,\frac{A}{B}=0$$

i.e.,

$$aB-2hH+bA=0$$

which is the required condition

Ex. 2. Lines are drawn through the origin perpendicular to the tangents from a point P to the ellipse $x^2/a^2+y^2/b^2=1$. Find the locus of P if these lines are conjugate diameters of the ellipse.

(*I. A. S.* 1972)

Solution. The equation of pair of tangents from the point P (x_1, y_1) to the ellipse is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

or

$$\frac{x^2}{a^2}\left(\frac{y_1^2}{b^2}-1\right)-\frac{2x_1y_1xy}{a^2b^2}+\frac{y^2}{b^2}\left(\frac{x_1^2}{a^2}-1\right)+\ldots\ldots=0; \quad \ldots(1)$$

The equation to the lines through the origin parallel to the lines given by equation (1) is

$$\frac{x^2}{a^2}\left(\frac{y_1^2}{b^2}-1\right)-\frac{2x_1y_1xy}{a^2b^2}+\frac{y^2}{b^2}\left(\frac{x_1^2}{a^2}-1\right)=0. \quad \ldots(2)$$

The equation of the lines through the origin perpendicular to the lines given by (2) is (see § 5.2, Q. 4)

$$\frac{x^2}{b^2}\left(\frac{x_1^2}{a^2}-1\right)+\frac{2x_1y_1xy}{a^2b^2}+\frac{y^2}{a^2}\left(\frac{y_1^2}{b^2}-1\right)=0.$$

These will be conjugate diameters of the ellipse (Example 1 above), if

$$\frac{1}{b^4}\left(\frac{x_1^2}{a^2}-1\right)+\frac{1}{a^4}\left(\frac{y_1^2}{b^2}-1\right)=0.$$

The locus of the point $P\,(x_1, y_1)$ is therefore

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=\frac{a^2}{b^2}+\frac{b^2}{a^2}.$$

Examples on Chapter VI

1. Find the pole of the line $6x+y+7=0$ with respect to the conic

$$x^2+2xy-y^2+2x+4y-1=0.$$

Ans. (2, 3).

2. Show that the tangent at an extremity of a diameter of a conic is parallel to the chords bisected by the diameter.

3. The two lines $x-\alpha=0$, $y-\beta=0$ are conjugate with respect to the hyperbola $xy=c^2$. Show that (α, β) lies on the conic $xy=2c^2$.

(*Lucknow*, 1978)

4. Show that the locus of the points such that the chords of contact of tangents drawn from them to the conic $ax^2+by^2=1$ subtend a right angle at the centre is the conic

$$a^2x^2+b^2y^2=a+b$$ (*Lucknow*, 1980)

5. Show that the line joining two points in the plane of a conic is the polar of the point of intersection of their polars with respect to the conic.

6. Show that the locus of the foot of the perpendicular from the centre of the ellipse $x^2/a^2+y^2/b^2=1$ on a tangent to it is

$$r^2=a^2\cos^2\theta+b^2\sin^2\theta.$$ (*Rajasthan*, 1973)

A9216

7. If the tangents drawn from an external point to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ be at right angles, show that the locus of their points of intersection is the circle

$$(ab-h^2)(x^2+y^2)-2(hf-bg)x-2(gh-af)y+c(a^2+b^2)-f^2-g^2=0.$$

Show also that this circle is concentric with the conic.

Note. This is known as *director circle*.

8. Show that the equation of tangents to the conic $x^2/a^2+y^2/b^2=1$ at the points of intersection with the line $lx+my+n=0$ is the conic

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)(a^2l^2+b^2m^2-n^2)=(lx+my+n)^2.$$

9. The polar of the point P with respect to the conic $y^2=4ax$ meets the curve in Q and R. Show that if P lies on the straight line $Ax+By+C=0$, the locus of the middle point of QR is the conic

$$A(y^2-4ax)+2a(Ax+By+C)=0. \qquad \text{(Lucknow, 1978)}$$

10. Find the locus of middle points of the chords of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

which are parallel to the line $lx+my+n=0$.

11. The ends of a chord are equidistant from a fixed point (x_0, y_0). Prove that the locus of the middle point of the chord is the conic.

$$(x-x_0)(hx+by+f)-(y-y_0)(ax+hy+g)=0.$$

11. Prove that the locus of the middle points of chords of the conic $x^2/y^2+y^2/b^2=1$ which subtend a right angle at the centre of the conic is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2.$$

(*Vikram*, 1967)

12. From a point $P(d\cos\theta, d\sin\theta)$, tangents PT, PT' are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Show that, as θ varies, the middle point of TT' describes the curve

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{d^2}.$$

13. Prove that the coordinates of the centre of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

are $\left(\frac{G}{C}, \frac{F}{C}\right)$, where $F\equiv gh-af$, $G\equiv fh-bg$, $C\equiv ab-h^2$.

If the diameter parallel to the tangent at P on the conic passes through the origin, prove that P lies on the diameter.

$$g(Cx-G)+f(Cy-F)=0.$$

(*Lucknow*, 1960; *Andhra*, 1963)

14. A conic $ax^2+by^2=1$ and a point P are given. Prove that the locus of a point Q whose polar makes an angle of 90° with PQ is a conic passing through P and the centre of the given conic.

15. Tangents as right angles are drawn to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Since that the locus of middle points of the chords of contacts is the curve

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2+b^2}. \qquad \text{(U. P. C. S., 1968)}$$

16. Show that the locus of the middle points of chords of constant length $2c$ of the conic $x^2/a^2+y^2/b^2=1$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)+\frac{c^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0.$$

17. Pair of tangents are drawn to the conic $\alpha x^2+\beta y^2=1$, so as to be always parallel to conjugate diameters of the conic

$$ax^2+2hxy+by^2=1.$$

Show that the locus of their point of intersection is the conic

$$ax^2+2hxy+by^2=\frac{a}{\alpha}+\frac{b}{\beta}. \qquad \text{(I. A. S., 1974)}$$

18. Two concentric conics have in general, one and only one pair of common conjugate diameters. Prove it.

Hint. Two concentric conics can be written as

$$ax^2+2hxy+by^2=1 \text{ and } a'x^2+2h'xy+b'y^2=1.$$

The diameters $Ax^2+2Hxy+By^2=0$ are conjugte with respect to both if

$$Ab-2Hh+Ba=0 \text{ and } Ab'-2Hh'+Ba'=0.$$

From these, the equation of common conjugate diameters is

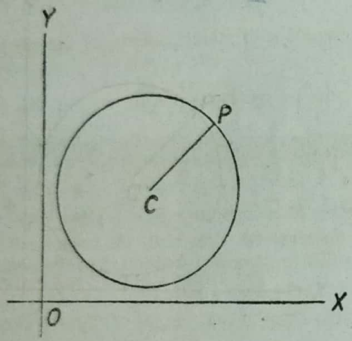
$$(ha'-ah')x^2-(ab'-a'b)xy+(bh'-b'h)y^2=0.$$

Scanned by CamScanner

CHAPTER VII

THE CIRCLE

7·1 Equation of a circle. Let (h, k) be the coordinates of the centre C of a circle whose radius is a.



If P is any point (x, y) on the circle, then $CP=a$.

But $$CP^2=(x-h)^2+(y-k)^2.$$

The equation of the circle whose centre is (h, k) and radius a is therefore

$$(\mathbf{x}-\mathbf{h})^2+(\mathbf{y}-\mathbf{k}^2)=\mathbf{a}^2. \quad ...(1)$$

If the axes, instead of being rectangular, were inclined at an angle ω, the equation of the above circle would have been

$$(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega=a^2.$$

We shall however work only in rectangular coordinates.

Corollary. The equation of the circle whose centre is the origin and radius a is

$$\mathbf{x}^2+\mathbf{y}^2=\mathbf{a}^2.$$

7·11 The general equation of the circle. Expanding the lefthand side of equation (1) of the preceding article, we have

$$x^2+y^2-2hx-2ky+h^2+k^2-a^2=0. \quad ...(1)$$

This is of the form

$$\mathbf{x}^2+\mathbf{y}^2+2\mathbf{gx}+2\mathbf{fy}+\mathbf{c}=0. \quad ...(2)$$

which is regarded as the *general equation* of a circle. Comparing coefficients in equations (1) and (2), we conclude that $-g=h$, $-f=k$ and $c=\sqrt{h^2+k^2-a^2}$. The *centre* of the circle represented by the general equation is therefore $(-\mathbf{g}, -\mathbf{f})$ and its *radius* is $\sqrt{\mathbf{g}^2+\mathbf{f}^2-\mathbf{c}}$.

If we compare equation (2) with the general equation of the second degree

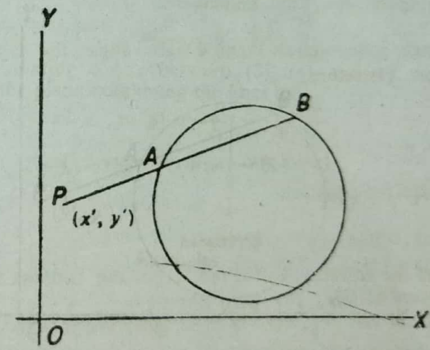
$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

we easily see that *the general equation of the second degree represents a circle if $a=b$ and $h=0$, that is, if the coefficients of x^2 and y^2 are equal and the coefficient of xy is zero.*

Note 1. Equation (2) represents a real or imaginary circle according as $g^2+f^2>$ or $<c$. If $g^2+f^2=c$, it represents a point circle.

Note 2. The general equation of a circle $x^2+y^2+2gx+2fy+c=0$ contains three constants g, f and c. If, therefore, three independent conditions satisfied by the constants are known, the equation of the circle is always determined.

7.12 The geometrical meaning of the constant c in the equation $\mathbf{x}^2+\mathbf{y}^2+2\mathbf{gx}+2\mathbf{fy}+\mathbf{c}=0$.



Let the straight line

$$\frac{x-x'}{l}=\frac{y-y'}{m}=r$$

through P, (x', y'), cut the circle $x^2+y^2+2gx+2fy+c=0$ in A and B. The distance PA and PB are the roots of the equation see) Chapter VI)

$$r^2(l^2+m^2)+2r\{(x'+g)l+(y'+f)m\}+x'^2-y'^2+2gx'+2fy'+c=0.$$

The product of the roots, *viz.*, $PA.PB$ is equal to

$$\frac{x'^2+y'^2+2gx'+2fy'+c}{\sqrt{l^2+m^2}}.$$

Since $l=\cos\theta$, $m=\sin\theta$, this is equal to $x'^2+y'^2+2gx'+2fy'+c$.

If $x'=0$ and $y'=0$, the $PA.PB=c$.

The constant c therefore represents the rectangle under the segments of chords of the origin.

7.13 Position of a point with regard to a circle. Let (x', y') be a given point and $x^2+y^2+2gx+2fy+c=0$ a given circle. If the point lies inside the circle, its distance from the centre is less than the radius; if it lies outside the circle, its distance from the centre is greater than the radius. Expressed analytically, *the point* (x', y') *lies inside or outside the circle accodring as* $(x'+g)^2+(y'+f)^2$ $<$ *or* $>$ g^2+f^2-c. That is, according as

$$x'^2+y'^2+2gx'+2fy'+c<0 \text{ or} >0.$$

7.14 Equation of the circle one of whose diameter is the line joining the points (x_1, y_1), (x_2, y_2). Let (x, y) be any point on the circle. The slopes of the straight lines joining (x, y) to (x_1, y_1) and (x_2, y_2) are respectively $\frac{y-y_1}{x-x_1}$ and $\frac{y-y_2}{x-x_2}$.

The angle in semi-circle being a right angle, the two lines are perpendicular. Therefore,

$$\frac{y-y_1}{x-x_1}\cdot\frac{y-y_2}{x-x_2}=-1,$$

or

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

which the required equation.

Examples

1. Find the equation to the circle passing through the points (0, 1), (1, 0) and (2, 1).

Ans. $x^2+y^2-2x-2y+1=0.$

2. Find the equations of the circles touching the straight lines $x+y=6$, $7x-y+42=0$ and $x-7x+42=0$.

Ans. $(2x+9)^2+(2y-9)^2=18$, $(x-2)^2+(y-12)^2=32$, $(x-12)^2+(y-2)^2=32$, $(x+3)^2+(y-3)^2=72$.

Hint. A circle touches a straight line if the perpendicular drawn from the centre on the line is equal to the radius.

3. Find the equation of the circle passing through the points (x_1, y_1), (x_2, y_2) and (x_3, y_3).

Solution. Let the required circle be

$$x^2+y^2+2gx+2fy+c=0. \qquad ...(1)$$

Since it passes through (x_1, y_1),

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0. \qquad ...(2)$$

Similarly, $x_2^2+y_2^2+2gx_2+2fy_2+c=0. \qquad ...(3)$

and $x_3^2+y_3^2+2gx_3+2fy_3+c=0. \qquad ...(4)$

Eliminating g, f and c from (1), (2), (3) and (4), the required equation is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0.$$

4. Whatever be the value of α, prove that the locus of the intersection of the straight lines $x\cos\alpha+y\sin\alpha=a$ and $x\sin\alpha-y\cos\alpha=b$ is a circle.

5. Find the equation to the circle which passes through the points (2, 3) and (4, 5), and of which the centre lies on the straight line $y-4x+3=0$

Ans. $x^2+y^2-4x-10y+25=0.$

6. *Find the equation of the circle circumscribing the triangle formed by the lines*

$$2y+x-5=0,\ y+2x-7=0,\ x-y+1=0.$$

Solution. The equation of the circle circumscribing the traingle formed by the given lines is

$$(2y+x-5)(y+2x-7)+\lambda(2y+x-5)(x-y+1)$$
$$+\mu(y+2x-7)(x-y+1)=0,$$

where λ and μ are so chosen that the coefficients of x^2 and y^2 are equal and the coefficient of xy is zero.

Simplifying, we obtain

$$x^2(2+\lambda+2\mu)+xy(5+\lambda-\mu)+y^2(2-2\lambda-\mu);$$
$$-x(17+4\lambda+5\mu)-y(19-7\lambda-8\mu)+35-5\lambda-7\mu=0.$$

Therefore,

$$2+\lambda+2\mu=2-2\lambda-\mu, \qquad ...(1)$$

and $5+\lambda-\mu=0. \qquad ...(2)$

From (1), $\lambda=-\mu \qquad ...(3)$

Therefore from (2) and (3), $\lambda=-5/2$ and $\mu=5/2$.

The equation to the circle therefore is

$$9\,(x^2+y^2)-39x-33y+60=0,$$

or

$$3\,(x^2+y^2)-13x-11y+20=0.$$

7. Find the equation to the circle circumscribing the triangle formed by the lines

$x+y=6,\ 2x+y=4$ and $x+2y=5$. (*Ranchi*, 1963)

Ans. $x^2+y^2-17x-19y+50=0$.

8. Show that the equation of the circle circumscribing the triangle formed by the lines $a_rx+b_ry+c_r=0,\ r=1, 2, 3$ is

$$\begin{vmatrix} \dfrac{1}{a_1x+b_1y+c_1} & a_2a_3-b_2b_3 & a_2b_3+b_2a_3 \\ \dfrac{1}{a_2x+b_2y+c_2} & a_3a_1-b_3b_1 & a_3b_1+b_3a_1 \\ \dfrac{1}{a_3x+b_3y+c_3} & a_1a_2-b_1b_2 & a_1b_2+b_1a_2 \end{vmatrix}=0.$$

9. Lines drawn through the points $(a, 0)$, $(-a, 0)$ make a constant angle α with one another. Show that the locus of their points of intersection are the circles

$$x^2+y^2-a^2\pm 2ay\cot\alpha=0.$$

Hence, show that the angles in the same segment of a circle are equal.

7.2 Equation of a tangent. From the result obtained for the general conic in Chapter VI we immediately see that the equation of the tangent at a point (x', y') of the circle

$$x^2+y^2+2gx+2fy+c=0$$

is

$$\mathbf{xx'+yy'+g\,(x+x')+f\,(y+y')+c=0.}$$

The slope of the tangent is $-\dfrac{x'+g}{y'+f}$.

The slope of the line joining the centre $(-g, -f)$ to (x', y') is $\dfrac{y'+f}{x'+g}$. The product of the two slope being equal to -1, *the tangent at a point is perpendicular to the radius through the point.* This property may conveniently be used to find the equation of the tangent to a given circle.

Corollary. The equation of the tangent at a point (x', y') of the circle $x^2+y^2=a^2$ is

$$\mathbf{xx'+yy'=a^2}.$$

7.21 Normal. *The normal at any point of a curve is the straight line drawn perpendicular to the tangent at that point.*

Since the equation of the tangent at a point (x', y') of the circle $x^2+y^2+2gx+2fy+c=0$ is $xx'+yy'+g\,(x+x')+f\,(y+y')+c=0$, i.e., $x\,(x'+g)+y\,(y'+f)+gx'+fy'+c=0$, *the equation of the normal at* (x', y') is

$$(x'+g)\,(y-y')-(y'+f)\,(x-x')=0$$

or

$$y\,(x'+g)-x\,(y'+f)+fx'-gy'=0.$$

Corollary. The equation of the normal at a point (x', y') of the circle $x^2+y^2=a^2$ is

$$xy'-yx'=0.$$

Note. Every normal to a circle passes through its centre.

7.22 Condition that the line $y=mx+c$ may be a tangent to the circle $x^2+y^2=a^2$.

Let (x', y') be the point of contact of the line with the circle.

Now, the equation of the tangent at (x', y') to the given circle is

$$xx'+yy'=a^2.$$

This being the same as the equation of the given line, we have,

$$\frac{x'}{-m}=y'-\frac{a^2}{c},$$

i.e.,

$$x'=-\frac{ma^2}{c},\ y'=\frac{a^2}{c}.$$

Since (x', y') is a points of the given circle,

$$x'^2+y'^2=a^2.$$

Substituting the values of x' and y' in the above we obtain

$$\frac{(1+m^2)\,a^4}{c^2}=a^2,$$

i.e.,

$$c=\pm a\sqrt{1+m^2},$$

which is the required condition.

Note. The value of c can be determined more easily by using the fact that the length of the perpendicular from the centre to the line $y=mx+c$ is equal to the radius of ... circle.

Corollary. The equation of a tangent to the circle $x^2+y^2=a^2$ is

$$\mathbf{y=mx+a\sqrt{1+m^2}},$$

and the coordinates of the point of contact are

$$\left(-\frac{\mathbf{am}}{\sqrt{\mathbf{1+m^2}}},\ \frac{\mathbf{a}}{\sqrt{\mathbf{1+m^2}}}\right).$$

Examples

1. Find the condition that the line $x \cos \alpha + y \sin \alpha - p = 0$ may touch the circle $x^2+y^2=a^2$.

Ans. $p = \pm a$.

2. Find the equation of the two tangents to $x^2+y^2=3$ which make an angle of $60°$ with the axis of x. (*Calcutta*, 1965)

Ans. $y = x\sqrt{3} \pm 2\sqrt{3}$.

3. Show that line $lx+my+n=0$ touches the circle $x^2+y^2+2gx+2fy+c=0$ if

$$(c-f^2)\, l^2+2fglm+(c-g^2)\, m^2-2n\,(gl+fm)+n^2=0.$$

4. Show that the line $x+y\sqrt{3}=4$ touches the circles $x^2+y^2=4$ and $x^2+y^2-4x-4\sqrt{3}y+12=0$ at the same point.

5. Find the equation so the circle whose centre is at the point (α, β) and which passes through the origin, and prove that the equation of the tangent at the origin is

$$\alpha x+\beta y=0.$$

6. Show that the circle

$$x^2+y^2+4x-4y+4=0$$

touches the axes of coordinates.

Find the equation of the tangents which make equal intercepts on the axes of coordinates.

Ans. $x+y=\pm 2\sqrt{2}$, $x-y+4+2\sqrt{2}=0$.

7. *The straight line* $px+qy=1$ *makes with*

$$ax^2+2pqxy+by^2=c$$

a chord which subtends a right angle at the origin. Show that $c\,(p^2+q^2)=a+b$, *and that the chord envelopes a circle whose radius is*

$$\sqrt{\frac{c}{a+b}} \qquad \text{(Lucknow, 1953)}$$

Solution. The pair of straight lines joining the origin with the intersections of the given line and the conic is

$$ax^2+2pxy+by^2=c\,(px+qy)^2.$$

i.e.,
$$x^2\,(a-cp^2)+2pxy\,(1-cq)+y^2\,(b-cq^2)=0.$$

These are perpendicular. Therefore,

$$a-cp^2+b-cq^2=0,$$

i.e.,
$$c\,(p^2+q^2)=a+b. \qquad ...(1)$$

Now, comparing $px+qy=1$ with the general equation $y=mx+a\sqrt{1+m^2}$ of the tangent to a circle, we have

$$\frac{p}{-m}=q=\frac{1}{a\sqrt{1+m^2}}$$

i.e
$$\frac{p}{q}=-m,$$

$$a=\frac{1}{q\sqrt{1+m^2}}=\frac{1}{\sqrt{p^2+q^2}}.$$

$$=\sqrt{\frac{c}{a+b}}$$

from relation (1). This proves the proposition.

8. *AP*, *BQ* are parallel tangents to a circle, and a tangent at any point *C* cuts them in *P* and *Q* respectively. Show that *CP.CQ* is independent of the position of the point *C*.

Hint. Two parallel tangents to the circle, $x^2+y^2=a^2$ are $x \cos \alpha + y \sin \alpha - a = 0$ and $x \cos \alpha + y \sin \alpha + a = 0$.

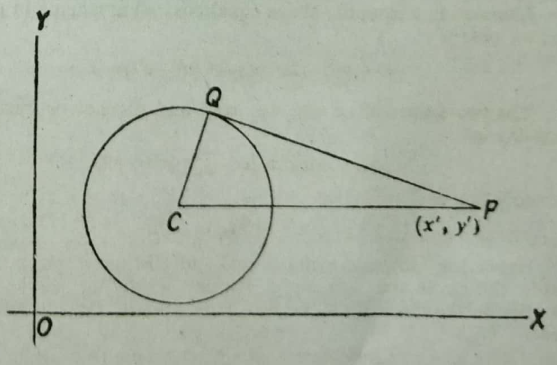
9. Find the locus of the feet of normals from (h, k) to the circles $x^2+y^2-2\lambda x=0$, where λ is variable.

Hint. The feet of the normals will be the points where the line joining (h, k) to the centre meets the circle. Eliminate λ from the equations

$$y-k=\frac{k}{h-\lambda}(x-h),$$

and
$$x^2+y^2-2\lambda x=0.$$

7.23. Length of the tangent from a given point. Let $x^2+y^2+2gx+2fy+c=0$ be a circle and let (x', y') be a point P outside the circle.



The coordinates of the centre C of the circle are $(-g, -f)$. If PQ is the tangent from P to the circle, the angle $CQP=90°$.

In the right angle triangle CPQ,

$$PQ^2=CP^2-CQ^2$$
$$=(x'+g)^2+(y'+f)^2-(g^2+f^2-c)$$
$$=x'^2+y'^2+2gx'+2fy'+c.$$

Hence the required length of the tangent is

$$\sqrt{x'^2+y'^2+2gx'+2fy'+c}$$

Corollary. The length of the tangent from (x', y') to the circle $ax^2+ay^2+2gx+2fy+c=0$ is

$$\sqrt{\left(x'^2+y'^2+\frac{2g}{a}x'+\frac{2f}{a}y'+\frac{c}{a}\right)}.$$

7.3 Pair of tangents. The equation to the pair of tangents from (x', y') to the circle $x^2+y^2=a^2$ (see Chapter VI) is

$$(x^2+y^2-a^2)(x'^2+y'^2-a^2)=(xx'+yy'-a^2)^2.$$

The coordinates of the points of contact satisfy the above equation and the equation

$$x^2+y^2-a^2=0$$

simultaneously.

The points of contact are therefore obtained by solving the simultaneous equations

$$xx'+yy'-a^2=0,$$

and $$x^2+y^2-a^2=0.$$

Eliminating y from the above equations and arranging in powers of x, we obtain

$$x^2(x'^2+y'^2)-2a^2xx'+a^2(a^2-y'^2)=0.$$

The two values of x will be real and distinct or imaginary according as

$$a^4x'^2 > \text{or} < a^2(a^2-y'^2)(x'^2+y'^2),$$

i.e., according as

$$y'^2(a^2-x'^2-y'^2) < \text{or} > 0.$$

Hence the tangents will be real and distinct if the point lies outside the circle and imaginary if it lies inside the circle. If the point lies on the circle itself the two tangents will be real and coincident.

Example. Find the equations of the tangent to the circle $x^2+y^2=25$ which pass through $(-1, 7)$, and show that they are at right angles.
(*I. I. T. Admission*, 1959)

Ans. $12x^2+7xy-12y^2-25x+175y-625=0$.

7.31 Chord of contact. The chord of contact of tangents from (x', y') to the circle $x^2+y^2=a^2$ (see Chapter VI) is

$$xx'+yy'=a^2.$$

If the point lies on the circle, the chord of contact coincides with the tangent at the point. If the point lies inside the circle the tangents from the point to the circle are imaginary, yet the chord of contact is a real line passing through the imaginary points of contact.

7.4 Pole and Polar. From Chapter VI, we see that the polar of the point (x', y'), called the pole, with respect to the circle

$$x^2+y^2+2gx+2fy+c=0$$

is $$xx'+yy'+g(x+x')+f(y+y')+c=0.$$

If the point lies on the circle the polar is the same as the tangent at the point.

Corollary. The polar is the point (x', y') with respect to the circle $x^2+y^2=a^2$ is $xx'+yy'=a^2$.

7.4.1 Properties of polar.

(*i*) *The polar is perpendicular to the line joining the centre to the pole.*

Let (x', y') be the point and $x^2+y^2=a^2$ the circle.

The equation of the polar (x', y') is

$$xx'+yy'=a^2 \qquad ...(1)$$

and the equation of the line joining the centre of the circle to the pole is

$$xy'-yx'=0. \qquad ...(2)$$

Lines (1) and (2) are obviously perpendicular.

(*ii*) *The pole and the point of intersection of the polar with the join of the centre and the pole are inverse points with regard to the circle.*

The inverse point of A with regard to a circle, centre C, is another point B such that $CB.\ CA=(\text{radius})^2$.

Using property (i), the distance of the point of intersection of the polar with the join of the centre and the pole is equal to the

length of the perpendicular from the centre on the polar. If therefore, (x', y') is the pole and $x^2+y^2=a^2$ the circle, this distance is equal to

$$\frac{a^2}{\sqrt{x'^2+y'^2}}.$$

Since the denominator is the distance of the pole from the centre, the proposition is proved.

(iii) *If the polar of a point P passes through another point Q then the polar of Q will pass through P.*

The result has already been proved in Chapter VI.

Example. Show that the points (x_1, y_1) and (x_2, y_2) are conjugate with respect to the circle $x^2+y^2=a^2$ if $x_1x_2+y_1y_2=a^2$.

Examples

1. Secants drawn from a given point P cut a given circle in points pairs A_1, A_2; $B_1, B_2, \ldots\ldots$; A_n, B_n. Show analytically that

$$PA_1.PB_1=PA_2.PB_2=\ldots=PA_n.PB_n=PT^2.$$

where PT is a tangent from P to the circle.

2. *P is the point* (a, b), *and Q is the point* (b, a). *Find the equation of the circle touching OP and OQ and P and Q, O being the origin.*

Solution. The equation to PQ is $x+y=a+b$. ...(1)

The equation to the chord of contact of tangents drawn from the origin to the circle

$$x^2+y^2+2gx+2fy+c=0$$

is

$$gx+fy+c=0. \qquad \text{...(2)}$$

Comparing coefficients of (1) and (2), we get

$$g=f=-\frac{c}{a+b}$$

Substituting these values in the equation of the circle, we obtain

$$x^2+y^2-\frac{2c}{a+b}(x+y)+c=0.$$

Since the circle passes through (a, b),

$$a^2+b^2+c,$$

which gives the equation of the circle.

3. From any point on the circle

$$x^2+y^2+2gx+2fy+c=0,$$

tangents are drawn to the circle

$$x^2+y^2+2gx+2fy+c\sin^2\alpha+(g^2+f^2)\cos^2\alpha=0.$$

Prove that the angle between them is 2α.

4. Tangents are drawn from the point (h, k) to the circle $x^2+y^2=a^2$. Prove that the area of the triangle formed by them and the straight line joining their points of contact is

$$\frac{a(h^2+k^2-a^2)^{3/2}}{h^2+k^2}. \qquad (\textit{Bihar}, 1963)$$

Hint. The length of each tangent is $\sqrt{h^2+k^2-a^2}$. If α be the angle between the two tangents, the area of the triangle is

$$\tfrac{1}{2}(h^2+k^2-a^2)\sin\alpha.$$

5. Show that the lines $lx+my+n=0$ and $l'x+m'y+n'=0$ are conjugate with respect to the circle $x^2+y^2=a^2$ if

$$(ll'+mm')a^2=nn'.$$

Hint. Use the property that conjugate lines are such that each contains the pole of the other.

6. Prove that the polar of the point (p, q) with respect to the circle $x^2+y^2=a^2$ touches $(x-c)^2+(y-d)^2=b^2$ if

$$b^2(p^2+q^2)=(a^2-cp-dq)^2. \qquad (\textit{Osmania}, 1960)$$

7. Prove that the distances of two points from the centre of a circle are proportional to the perpendiculars drawn from one point on the polar of the other.

Hint. Take the equation of the circle as $x^2+y^2=a^2$.

8. Show that the locus of the poles of the line $lx+my+n=0$ with respect to circle which touch the x-axis at the origin is

$$y(mx-ly)=nx.$$

Hint. The general equation of the circle touching the x-axis at the origin is $x^2+y^2+2ky=0$.

9. Find the locus of the poles of the line

$$\frac{x}{a}+\frac{y}{b}=1$$

with respect to the circles which touch the coordinate axes

Ans. $(ax-by)(ay-bx)+ab(a\pm b)(x\pm y)=0$.

7.5 Equation to the chord of the circle $x^2+y^2=a^2$ whose middle point is (x', y').

As in Chapter VI, the equation to the chord of the circle $x^2+y^2=a^2$ whose middle point is (x', y') is

$$xx'+yy'-a^2=x'^2+y'^2-a^2,$$

or

$$\mathbf{xx'+yy'=x'^2+y'^2}.$$

Note. The required equation can also be obtained by using the property that the perpendicular from the centre upon any chord bisects it. The student should do it as an exercise.

Examples

1. *Prove that the locus of the middle points of a series of parallel chords of a circle, is a straight line passing through the centre.*

Solution. Let the circle be $x^2+y^2=a^2$ and let (x', y') be middle point of a chord drawn parallel to $y=mx+c$.

The equation to the chord is $xx'+yy'=x'^2+y'^2$.

The slope of this chord is $-\dfrac{x'}{y'}=m$. Hence the locus of (x', y') is $my+x=0$ which passes through the centre of the circle.

2. Find the middle point of the chord of the circle $x^2+y^2=a^2$ lying along the line $lx+my=n$.

Ans. $\left(\dfrac{ln}{l^2+m^2}, \dfrac{mm}{l^2+m^2}\right)$.

Hint. Compare $xx'+yy'=x'^2+y'^2$ and $lx+my=n$.

3. Find the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which pass through the fixed point (h, k).

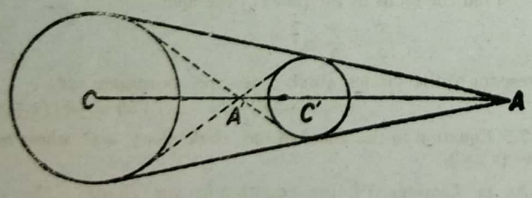
(*Rajsthan*, 1965)

Ans. $x^2+y^2-xh+yk=0$.

4. Show that the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which subtend a right angle at $(c, 0)$ is the circle

$$2(x^2+y^2)-2cx+c^2-a^2=0 \qquad (\textit{Vikram}, 1964)$$

7.6 Common tangents to two circles.



We shall use the following results of pure geometry.

(i) Two direct common tangents and two transverse common tangents can be drawn to two circles which are such that one lies wholly outside the other. In case of external contact of two circles the two transverse common tangents coincide and only three common tangents can be drawn.

(ii) If two circles cut one another in real points, only the two direct common tangents can be drawn. In the limiting case of internal contact of two circles the two direct common tangents coincide and only one common tangent can be drawn to two circles.

(iii) The two direct common tangents meet the line of centres CC' in the same point A and the two transverse common tangents also meet CC' in the same point A'. A and A' which are called the *centres of similitude* divide CC' externally and internally in the ratio of the radii of the circles.

Examples

1. *Find the equation of the common tangents of the circles*

$$x^2+y^2-24x+2y+120=0,$$

and

$$x^2+y^2+20x-6y-116=0. \qquad (\textit{Lucknow}, 1959)$$

Solution. The centres of the two circles are $(12, -1)$ and $(-10, 3)$ and their radii are $\sqrt{12^2+1^2-120}$ and $\sqrt{10^2+3^2+116}$ *i.e.*, 5 and 15. The circles are such that one lies wholly outside the other.

The points dividing the line of centres externally and internally in the ratio 5 : 15 are $(23, -3)$ and $(\frac{1}{2}{}^{3}, 0)$.

The equation to any line through $(23, -3)$, is

$$y+3=m(x-23).$$

This touches the first circle if

$$\frac{11m+2}{\sqrt{1+m^2}}=\pm 5,$$

i.e., if
$$96m^2+44m-21=0,$$

from which,
$$(4m+3)(24m-7)=0$$

i.e.,
$$m=-\frac{3}{4} \text{ or } m=\frac{7}{24}.$$

The equations to the two *direct common tangents* are therefore

$$4x+4y=57 \text{ and } 7x-24y=233.$$

Similarly, the equation to any line through $\left(\frac{13}{2}, 0\right)$ is

$$2y=m(2x-13),$$

This touches the circle if

$$\frac{11m+2}{2\sqrt{1+m^2}}=\pm 5$$

i.e., if $21m^2+44m-96=0$

from which $m=\frac{4}{3}$ or $-\frac{7}{24}$.

The *transverse common tangents* are therefore

$$3y-4x+26=2 \text{ and } 7y+24x-156=0.$$

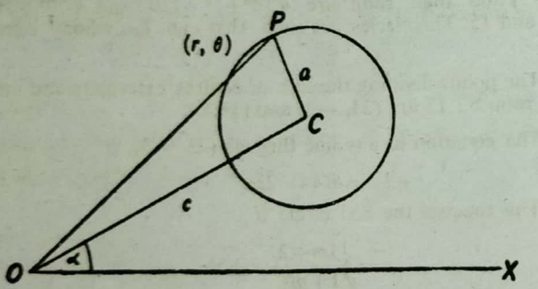
2. Find the common external tangents to the two circles $x^2+y^2=16$ and $x^2+y^2+6x-8y=0$. (*Delhi*, 1945)

Ans. $6x+4y-8=\pm\sqrt{6}\,(x-12)$.

3. Prove that the two circles $x^2+y^2+2ax+c=0$ and $x^2+y^2+2by+c=0$ touch, if $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c}$. (*U. P. C. S.*, 1971)

4. Show that the common tangents of the circles $x^2+y^2+2x=0$ and $x^2+y^2-6x=0$ form an equilateral triangle. (*Punjab*, 1964)

7.7 Equation of a circle in polar coordinates.



Let C be the centre (c, α) and a the radius of a given circle. Let P be any point (r, θ) on the circle. Then, in the $\triangle\, OPC$,

$$PC^2=OC^2+OP^2-2OC.OP\cos OPC$$

i.e., $a^2=c^2+r^2-2cr\cos(\theta-\alpha)$.

The equation of the given circle is therefore

$$\mathbf{r^2-2cr\cos(\theta-\alpha)+c^2-a^2=0.}$$

Corollary 1. The equation of a circle passing through the pole is $r=2a\cos(\theta-\alpha)$. For, here $c=a$,

Corollary 2. The equation of a circle passing through the pole and having its centre on the initial line is $r=2a\cos\theta$. For, here $c=a$ and $\alpha=0$.

Examples

1. Find the polar equation of the circle described on the straight line joining the points (r_1, θ_1) and (r_2, θ_2) as diameter.

Ans. $r^2-r\,[r_1\cos(\theta-\theta_1)+r_2\cos(\theta-\theta_2)]+r_1r_2\cos(\theta_1-\theta_2)=0$.

2. Show that the straight line $\frac{1}{r}=A\cos\theta+B\sin\theta$ touches circle $r=2a\cos\theta$, if $a^2B^2+2aA=1$. (*Lucknow*, 1960)

3. The vectorial angles of two points P and Q on the circle $r=2a\cos\theta$ are θ_1 and θ_2. Find the equation to PQ. Hence deduce the equation of the tangent at P. (*Punjab*, 1976)

Ans. $r\cos(\theta-\theta_1-\theta_2)=2a\cos\theta_1\cos\theta_2$.

Hint. Any line in polar coordinates is $p=r\cos(\theta-\alpha)$. Determine p and α so that it passes through P and Q.

4. *A circle passes through the point (r_1, θ_1) and touches the initial line at a distance from the pole. Show that its polar equation is*

$$\frac{r^2-2cr\cos\theta+c^2}{r\sin\theta}=\frac{r_1^2-2cr_1\cos\theta_1+c^2}{r_1\sin\theta_1}$$

(*Agra*, 1960)

Solution. If a be the radius of the circle, the coordinates of its centre are $\sqrt{a^2=c^2}, \alpha)$ where $\tan\alpha=\frac{a}{c}$. The equation to the circle is

$$r^2-2\sqrt{a^2+c^2}.\,r\cos(\theta-\alpha)+c^2=0,$$

or $r^2-2r\,(c\cos\theta+a\sin\theta)+c^2=0$,

i.e., $r^2-2cr\cos\theta+c^2=2ar\sin\theta$. ...(1)

Since it passes through (r_1, θ_1),

$$r_1^2-2cr_1\cos\theta+c^2=2ar_1\sin\theta_1. \quad ...(2)$$

Equating the value of a from (1) and (2), the equation to the circle becomes

$$\frac{r^2-2cr\cos\theta+c^2}{r\sin\theta}=\frac{r_1^2-2cr_1\cos\theta_1+c}{r_1\sin\theta_1}.$$

7.8 Parametric equation of a circle. Since the coordinates of any point on the circle $x^2+y^2=a^2$ can be expressed as $(a\cos\theta, a\sin\theta)$, the circle can be regarded as the locus of a point which moves such that its abscissa is $a\cos\theta$ and ordinate $a\sin\theta$, where θ is a

variable. We say that the equation to the circle in terms of the parameter 'θ' is

$$x=a\cos\theta,$$
$$y=a\sin\theta.$$

The point $(a\cos\theta, a\sin\theta)$ is, for brevity, called the point 'θ'.

Note. θ is the angle which the line joining the point 'θ' to the centre makes with the x-axis.

Example. Show that the parametric equation

$$x=\frac{a(1-t^2)}{1+t^2},\ y=\frac{2at}{1+t^2}$$

represents a circle of radius a.

Hint. Put $t=\tan\theta$ and eliminate θ.

Examples on Chapter VII

1. Find the equation to the straight line which passes through $x=1, y=1$ and $x=3, y=2$.

What is the equation of the circle of which this line is a tangent and whose centre is the origin?

Ans. $x-2y+1=0$; $5x^2+5y^2=1$.

2. Show that the locus of a point such that the ratio of its distances from two given points is constant, is a circle.

3. Find the length of the chord joining points in which the straight line $\frac{x}{a}+\frac{y}{b}=1$ meets the circle $x^2+y^2-r^2=0$.

Ans. $2\left\{\frac{r^2(a^2+b^2)-a^2b^2}{a^2+b^2}\right\}^{1/2}$.

4. Show that the four points of intersection of the lines $(2x-y+1)(x-2y+3)=0$ with coordinate axes lie on a circle. Find its centre and radius. (*Delhi*, 1962; *Agra*, 1965)

Ans. Centre : $\left(-\frac{7}{4}, \frac{5}{4}\right)$, *Radius* $\left(\frac{5\sqrt{2}}{4}\right)$.

5. Find the equation of the tangents to the circle

$$x^2+y^2-6x+4y=12$$

which are parallel to the line $3x+3y+5=0$. (*Roorkee*, 1966)

6. Find the equation of a circle which touches the axis of y and passes through two points on the axis of x on the same side of the origin.

Two circles touch the axis of y and intersect in the points $(1, 0)$, $(2, -1)$. Find their radii and show that they both touch the line $y+2=0$.

Ans. (1, 5).

7. Find the equations to the two circles whose centres are (0, 0) and (2, 0) and whose radii are equal to 2. Also find the equation to the line joining their common points.

Ans. $x=1$.

8. Find the equations to the two circles of which the centre lies on the line $4x+3y-2=0$ and which touch the straight lines $x+y+4=0$ and $y=7x+4$. (*I. I. T. Admission*, 1961)

Ans. $x^2+y^2-4x+4y=0$, $x^2+y^2+8x-12y+34=0$.

9. Find the length of the common chord of the circles

$$(x-a)^2+y^2=a^2 \text{ and } x^2+(y-b)^2=b^2$$

and prove that the equation of the circle whose diameter is this common chord is

$$(a^2+b^2)(x^2+y^2)=2ab(bx+ay).$$

(*Gorakhpur*, 1965; *Roorkee*, 1962; *U. P. C. S.*, 1976)

10. Prove that the circles

$$x^2+y^2-6x-2y+1=0 \text{ and } x^2+y^2+2x-8y+13=0$$

touch each other and find the equation of the tangent of the point of contact.

Ans. $4x-3y+6=0$.

11. Show that the circle on the chord

$$x\cos\alpha+y\sin\alpha-p=0$$

of the circle $x^2+y^2=a^2$ as diameter is

$$x^2+y^2-a^2-2p(x\cos\alpha+y\sin\alpha-p)=0.$$

(*Andhra*, 1962; *Lucknow*, 1962; *Roorkee*, 1967)

12. If $y=mx$ be the equation of a chord of a circle $x^2+y^2-2ax=0$, prove that the equation of the circle of which this chord is the diameter, is

$$(1+m^2)(x^2+y^2)-2a(x+my)=0.$$ (*Vikram*, 1967)

13. Prove that the tangent to the circle $x^2+y^2=5$ at the point $(1, -2)$ also touches the circle $x^2+y^2-8x+6y+20=0$, and find the coordinates of the point of contact. (*I. I. T. Admission*, 1975)

Ans. (3, −1).

14. Find the equation to the circle whose diameter is the common chord of the circles

$$x^2+y^2+2x+3y+1=0 \text{ and } x^2+y^2+4x+3y+2=0.$$

(*Andhra*, 1963; *Delhi*, 1963)

Ans. $2(x^2+y^2)+2x+6y+1=0$.

15. Prove that the polar of a given point with respect to the circles $x^2+y^2-2kx+c^2=0$ always passes through a fixed point whatever be the value of k.

16. Find the equation to the circle which touches the coordinate axes and the line $x/a+y/b=1$, the centre lying in the first quadrant.

(*Vikram*, 1968).

$$Ans.\ x^2+y^2-\left(\frac{ab}{a+b}\right)(x+y)+\frac{a^2b^2}{A\,(a+b)^2}=0.$$

17. Show that the circum circle of the traingle formed by the lines $ax+by+c=0$, $bx+cy+a=0$ and $cx+ay+b=0$ passes through the origin if

$$(b^2+c^2)\,(c^2+a^2)\,(a^2+b^2)=abc\,(a+b)\,(b+c)\,(c+a).$$

18. Show that the straight line $x+y-1=0$ is a common tangent to the circles $2\,(x^2+y^2-2x-2y)+3=0$ and $2x^2+2y^2-12x+4y+19=0$.

19. Show that the locus of a point which moves such that the sum of the squares of its distances from the vertices of a triangle is constant, is a circle whose centre is the centroid of the triangle.

(*Andhra*, 1960)

20. P is a point on the circle $x^2+y^2+2x+6y-15=0$; Q is a point on the line $7x+y+3=0$, and the perpendicular bisectors of PQ is the line $x-y+1=0$. Show that there are two positions of the point P and find their coordinates.

Ans. $(3, 0)$, $(-4, 1)$.

21. The circle $x^2+y^2=a^2$ cuts off an intercept on the straight line $lx+my=1$, which subtends an angle of 45° at the origin. Show that

$$4\,[a^2\,(l^2+m^2)-1]=[a^2\,(l^2+m^2)-2]^2.$$

22. The distances from the origin of centres of three circles $x^2+y^2-2\lambda x=c^2$ (where c is constant and λ variable) are in geometric progression; prove that the lengths of the tangents drawn to them from any point on the circle $x^2+y^2=c^2$ are also in geometric progression.

23. Show that the locus of the middle points of the chords of contact of tangents drawn to a given circle from points on another given circle is a third circle.

24. Find the locus of the middle point of the chords of a circle which are of constant length

Ans. A circle.

25. Given the base of a triangle and the ratio of the lengths of the other two unequal sides, prove that the vertex lies on a fixed circle. (*I. I. T. Admission*, 1973)

26. The lengths of tangents from two points A and B to a circle are l and l' respectively. If the points are conjugate with respect to the circle, show that $AB^2=l^2+l'^2$.

27. Circles are drawn passing through the point $(0, k)$ and touching the circle $x^2+y^2=a^2$. Find the locus of the pole of the axis of y with respect to these circles.

$$Ans.\ 4a^2\,(y-k)^4=(a^2-k^2)\,[a^2-(k-2y)^2]\,x^2.$$

28. Show that the equation of the circle to which the traingle whose vertices are (x_r, y_r), $r=1, 2, 3$ is self-conjugate if

$$\begin{vmatrix} x^2+y^2 & 2x & 2y & 1 \\ x_1x_2+y_1y_2 & x_1+x_2 & y_1+y_2 & 1 \\ x_2x_3+y_2y_3 & x_2+x_3 & y_2+y_3 & 1 \\ x_3x_1+y_3y_1 & x_3+x_1 & y_3+y_1 & 1 \end{vmatrix}=0.$$

29. O is a fixed point and P is any point on a given straight line. OP is joined and a point Q is taken on it such that $OP.OQ=$constant. Show that locus of Q is a cirele which passes through O.

(*Lucknow*, 1961)

Hint. Use polar coordinates.

30. Find the coordinates of the vertices of a rhombus, the sides of which are tangential to the circle $x^2+y^2=2x$. Its longer side is $\sqrt{2}$ times the smaller one in length and is parallel to y-axis. Also show that the area of the rectangle obtained by joining the points of contact is $(4\sqrt{2}/3)$ square units

Ans. $(1, \pm\sqrt{3})$, $(1\pm\sqrt{\tfrac{3}{2}}, 0)$.

31. If $x\cos\theta+y\sin\theta=2$ is the equation of a common tangent to $x^2+y^2=4$ and $x^2+y^2=6\sqrt{3}x-6y+20=0$, find the value of θ. Prove also that the angle between the pair of direct common tangents is $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$.

$$Ans.\ \frac{1}{6}\pi, \text{ or } \frac{1}{6}\pi\pm\cos^{-1}\left(-\frac{1}{3}\right).$$

32. Prove that the orthocentre of the triangle whose angular points are $(a\cos\alpha_r, a\sin\alpha_r)$, $r=1, 2, 3$ is the point

$$\left[a\sum_{r=1}^{3}\cos\alpha_r,\quad a\sum_{r=1}^{3}\sin\alpha_r\right].$$

Hence prove that the centroid of any triangle divides the join of the circumcentre and orthocentre in the ratio 1 : 2.

Hint. The circumcentre is the origin.

33. Find the equation of the circle circumscribing the traingle formed by the lines

$$ax^2+2hxy+by^2=0 \text{ and } x\cos\alpha+y\sin\alpha-p=0.$$

Also find the area of the triangle.

Ans. $(x^2+y^2)(a\sin^2\alpha-2h\sin\alpha\cos\alpha+b\cos^2\alpha)-p\,[(a-b)\sin\alpha$
$-2h\cos\alpha]\,x+py\,[(a-b)\cos\alpha+2h\sin\alpha]=0;$
$[(p^2\sqrt{h^2-ab})/(a\sin^2\alpha-2P\sin\alpha\cos\alpha+b\cos^2\alpha)].$

34. Find the equation to the circles which passes through the point (2, 0) and whose centre is the limit of the point of intersection of the lines

$$3x+5y=1$$
$$(2+c)\,x+5\,c^2y=1 \text{ as } c \text{ tends to } 1.$$

(*I. I. T. Admission*, 1979)

Ans. $25(x^2+y^2)-20x+2y-60=0.$

CHAPTER VIII

SYSTEMS OF CIRCLES

8.1 Orthogonal intersection of two circles. *The angle at which two curves intersect at a point is defined to be the angle between the tangents to the curves at the point.*

Two curves are said to intersect *orthogonally* when the tangents at the common point, are at right angles.

We shall find the necessary and sufficient condition for the orthogonal intersection of the circles

$$x^2+y^2+2gx+2fy+c=0,$$

and
$$x^2+y^2+2g'x+2f'y+c'=0.$$

The centres of the given cirles are $(-g,\ -f)$ and $(-g',-f')$ and their respective radii are $\sqrt{g^2+f^2-c}$ and $\sqrt{g'^2+f'^2-c'}$.

Now, since the radius through a point of a circle is perpendicular to the tangent at that point, it follows that the angle of intersection of the circle is the same as the angle between their radii drawn through the points of intersection. In the case of orthogonal intersection, this angle is 90° and consequently the square of the distance between the centres of the circles is equal to the sum of the squares of their radii.

For orthogonal intersection we thus have

$$(g-g')^2+(f-f')^2=g^2+f^2-c+g'^2+f'^2-c'.$$

i.e.,
$$2gg'+2ff'=c+c'.$$

This is therefore the *necessary* condition for orthogonal intersection of two circles. That the condition is *sufficient* also can be seen by working the algebra backwards.

Examples

1. *Prove that the two circles which pass through two points* $(0, a)$ *and* $(0, -a)$ *and touch the straight line* $y=mx+c$, *will cut orthogonally if* $c^2=a^2(2+m^2)$. (*Andhra*, 1962; *Gorakhpur*, 1964)

Solution. The centre of each circle lies on x-axis. If, therefore, $(k, 0)$ are the coordinates of the centre of one of the circles, its radius is $\sqrt{k^2+a^2}$ and since it touches the given line, we have

$$\frac{mk+c}{\sqrt{1+m^2}}=\pm\sqrt{k^2+a^2}$$

i.e., $$k^2-2mck+a^2(1+m^2)-c^2=0. \quad \ldots(1)$$

Let k_1, k_2 be the roots of (1). The equations of the two circles are then

$$x^2+y^2-2k_1k-a^2=0,$$

and $$x^2+y^2-2k_2x-a^2=0.$$

These will cut orthogonally if

$$2k_1k_2=-2a^2. \quad \ldots(2)$$

From (1), $\quad k_1k_2=a^2(1+m^2)-c^2.$

Therefore from (2),

$$a^2(2+m^2)=c^2.$$

2. *Obtain the equation of the circle which cuts orthogonally the circle $x^2+y^2-6x+4y-3=0$, passes through* (3, 0) *and touches the axis of y.* (*Delhi*, 1964)

Solution. The general equation of a circle which touches the axis of y is

$$x^2+y^2-2hx-2ky+k^2=0.$$

If it passes through (3, 0)

$$k^2=6h-9. \quad \ldots(1)$$

If it cuts the given circle orthogonally,

$$6h-4k=k^2-3. \quad \ldots(2)$$

From (1) and (2), $k=3$ and $h=3$.

The required equation is therefore

$$x^2+y^2-6x-6y+9=0.$$

3. Find the equation to the circle which passes through (1, 1) and cuts orthogonally each of the circles

$$x^2+y^2-8x-2y+16=0 \text{ and } x^2+y^2-4x-4y-1=0.$$

Ans. $3x^2+3y^2-14x+23y-15=0.$

4. Show that if a circle which cuts $x^2+y^2=c^2$ orthogonally, passes through the point (α, β), it will also pass through the point

$$\left(\frac{c^2\alpha}{\alpha^2+\beta^2}, \frac{c^2\beta}{\alpha^2+\beta^2}\right) \quad (\textit{Rajasthan}, 1964)$$

Hint. The general equation of the circle cutting the given circle orthogonally, is

$$x^2+y^2+2gx+2fy+c^2=0.$$

5. If the equation of two circles whose radii are a, a' be $S=0$, $S'=0$ then show that the circles

$$\frac{S}{a} \pm \frac{S'}{a'}=0$$

will intersect at right angles. (*Rajasthan*, 1964)

8.2 Radical axis. *Definition. The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the circles are equal.*

Let us find the equation to the radical axis of the circles whose equations are

$$x^2+y^2+2gx+2fy+c=0$$

and $$x^2+y^2+2g'x-2f'y-c'=0.$$

If (x', y') is a point on the radical axis, the lengths of the tangents from (x', y') to these circles must be equal. We then have

$$x'^2-y'^2-2gx'+2fy'+c=x'^2+y'^2-2g'x'+2f'y'+c',$$

i.e., $$2(g-g')x'+2(f-f')y'+c-c'=0.$$

The locus of (x', y'), or the equation to the radical axis of the given circle is, therefore,

$$\mathbf{2(g-g')x+2(f-f')y+c-c'=0.}$$

Note 1. *The radical axis of two circles is a straight line.*

Note 2. If the equations of two circles are written such that the coefficients of x^2 and y^2 in each are unity and the equations so obtained denoted by $S=0$ and $S_1=0$, the equation to their radical axis is $S-S_1=0$. But $S-S_1=0$ passes through the points of intersection of $S=0$ and $S_1=0$. The radical axis of two circles is therefore the same as their *common chord.* If the circles touch each other the radical axis is their common tangent at the point of contact.

8.2.1 Some theorems on radical axis. We shall give below a few theorems connected with the radical axis.

Theorem I. *The radical axis of two circles is perpendicular to the line joining their centres.*

Let the two circles be

$$x^2+y^2+2gx+2fy+c=0$$

and $$x^2+y^2+2g'x+2f'y+c'=0.$$

The centres of the circles are $(-g, -f)$, $(-g', -f')$.

The equation to the line of centres is

$$y+f=\frac{f-f'}{g-g'}(x+g).$$

Also, the radical axis is

$$2(g-g')x+2(f-f')y+c-c'=0.$$

The product of the slopes of these lines is

$$\frac{f-f'}{g-g'}\times\left(-\frac{g-g'}{f-f'}\right)=-1,$$

which shows that the radical axis is perpendicular to the line of centres.

Theorem II. *The radical axis of three circles, taken in pairs meet in a point.*

Let the equations of three circles be

$$S=0,\ S_1=0,\ S_2=0.$$

the coefficients of x^2 and y^2 being unity in each equation.

The radical axes of the circles, taken in pairs, are

$$S-S_1=0,$$
$$S_1-S_2=0,$$
$$S_2-S=0.$$

Adding the left-hand sides, we find that their sum identically vanishes. The three lines represented by these equations are therefore concurrent.

The common point of the radical axes of three circles, taken in pairs, is called the *Radical Centre.*

Theorem III. *The difference of the squares of the tangents to two circles from any point in their plane varies as the distance of the point from their radical axis.*

If we take the line of centres of the circles as the axis of x, and the radical axis as the axis of y, the equations of the circles are written as

$$x^2+y^2+2gx+c=0$$

and

$$x^2+y^2+2g'x+c=0.$$

For the radical axis is $2(g-g')x=0$, *i.e.*, $x=0$ and the centres are $(-g, 0)$, $(-g', 0)$, which evidently fulfils the conditions for special choice of axes.

Now, let (x_1, y_1) be any point in the plane of the circles.

The difference of the squares of the tangents drawn from (x_1, y_1) to the circles is equal to

$$x_1^2+y_1^2+2gx_1+c-x_1^2-y_1^2-2g'x_1-c=2(g-g')x_1,$$

which varies as x_1, which is the distance of (x_1, y_1) from the axis of y, that is the radical axis.

Theorem IV. *If two circles cut a third circle orthogonally, the radical axis of the two circles passes through the centre of the third circle.* (*Lucknow*, 1968)

Choosing the line of centres of two circles as the axis of x, and their radical axis as the axis of y, the equations of the circles are

$$x^2+y^2+2gx+c=0, \qquad ...(1)$$

and

$$x^2+y^2+2g'x+c=0. \qquad ...(2)$$

Let $\quad x^2+y^2+2Gx+2Fy+C=0. \qquad ...(3)$

be a third circle which is cut orthogonally by (1) and (2).

We then have

$$2Gg+2F.0=c+C,$$

and

$$2Gg'+2F.0=c+C,$$

From these, $\quad G=0,\ C=-c,$

and equation (3) reduces to

$$x^2+y^2+2Fy-c=0. \qquad ...(4)$$

The centre of (4) is $(0, -F)$, which lies on the axis of y or the radical axis of (1) and (2).

Corollary. The radical centre of three circles is the centre of the circle which cuts them orthogonally, and the radius of this fourth circle is equal to the length of the tangent from the radical centre of anyone of the three given circles.

Theorem V. *The radical axis of two circles bisects each of their common tangents.*

Theorem VI. *The radical centre of three circles described on the sides of a triangle as diameters, is the orthocentre of the triangle.*

The proofs of the last two theorems are left as exercises to the student.

8.3 The equations $S+\lambda S_1=0$, $S+\lambda u=0$.

If $S=0$ and $S_1=0$ be two circles, $S+\lambda S_1=0$, where λ is a constant, is also a circle, for the coefficient of x^2 and y^2 in $S+\lambda S_1=0$ are equal and the coefficient of xy is zero. Further the values of x and y which simultaneously satisfy $S=0$, $S_1=0$, obviously satisfy $S+\lambda S_1=0$.

Hence $S+\lambda S_1=0$ *is a circle which passes through the point of intersection of the circles* $S=0$, $S_1=0$.

If $S=0$, $S_1=0$ are two conics, $S+\lambda S_1=0$ is a conic which passes through the points of intersection of $S=0$, $S_1=0$.

Again, if $u=0$ is the equation to a straight line and $S=0$ the equation to a circle, $S+\lambda u=0$ is a circle since in this equation

the coefficients of x^2 and y^2 are equal, and there is no term in xy. This circle passes through the points of intersection of $S=0$ and $u=0$.

Corollary. 1. If $S=0$ and $S_1=0$ be the equations of two circles the equations of any two circles having the same radical axis as $S=0$ and $S_1=0$ are

$$S+\lambda_1 S_1=0,\ S+\lambda_2 S_1=0.$$

Corollary. 2. $S+\lambda u=0$ is the equation of a circle such that the radical axis of it and the circle $S=0$ is the line $u=0$.

Examples

1. *Find the coordinates of the radical centre of the three circles*

$$x^2+y^2=9,\ x^2+y^2-2x-2y=5 \text{ and } x^2+y^2+4x+6y=19.$$

Solution. The radical axis of the first two circles is $x+y=2$, and that of the first and the third circles is $2x+3y=5$. The coordinates of the radical centre are therefore (1, 1).

2. Find the general equation of the system of circles any pair of which have the same radical axis as the circles

$$x^2+y^2+x-5y-3=0,$$

and

$$x^2+y^2+3x+4y+6=0.$$

Show that the equation to that member of the system which passes through the origin is

$$3x^2+3y^2+5x-6y=0.$$

3. Show that the locus of points such that the difference of the squares of the tangents from them to two given circles is constants, is a line parallel to their radical axis.

4. Two circles intersect in the point $A:(x_1, y_1)$ and the line joining the other extremities of the two diameters through A, makes an angles θ with the axis of x. Prove that the equation of radical axis of the circles is

$$(x-x_1)\cos\theta+(y-y_1)\sin\theta=0.$$

Hint. The radical axis passes through (x_1, y_1) and is perpendicular to the line joining the other extremities of the two diameters through A.

5. Find the equation to the circle which cuts orthogonally each of the three circles

$$x^2+y^2+2x+17y+4=0,$$
$$x^2+y^2+7x+6y+11=0,$$
$$x^2+y^2-x+22y+3.$$

(*Delhi*, 1960)

Ans. $x^2+y^2-6x-4y-44=0$.

6. *If the four points in which the two circles*

$$x^2+y^2+ax+by+c=0,\ x^2+y^2+a'x+b'y+c'=0$$

are intersected by the straight lines

$$Ax+By+C=0,\ A'x+B'y+C'=0$$

respectively, lie on another circle, then prove that

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix}=0.$$

(*Allahabad*, 1964; *Gorakhpur*, 1965; *Patna*, 1968)

Solution. The radical axis of the two given circles is

$$(a-a')\,x+(b-b')\,y+c-c'=0. \qquad ...(1)$$

The radical axis of the third circle and the two given circles are respectively

$$Ax+By+C=0, \qquad ...(2)$$

and

$$A'x+B'y+C'=0. \qquad ...(3)$$

The lines (1), (2) and (3) have a common point of intersection. Therefore, eliminating x and y, we have

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix}=0.$$

8.4 Coaxal circles. *Definition.* A system of circles is said to be coaxal when every pair of the circles has the same radical axis.

The following results follow from the above definition.

(i) The centres of all circles of a coaxal system lie in one straight line which is perpendicular to the common radical axis because the line of centres of any pair of circles cuts the radical axis at right angles.

(ii) Circles passing through two fixed points form a coaxal system, for every pair of circles has the same common chord and therefore the same radical axis.

(iii) The equation to a coaxal system of which two members are $S=0$ and $S_1=0$ is $S+\lambda S_1=0$ where λ is a constant.

(iv) The equation to a coaxal system of which one member is the circle $S=0$ and of which the common radical axis is the line $u=0$ is $S+\lambda u=0$ where λ is a constant.

8.41 Equation to a coaxal system. Let us find the equation to a system of coaxal circles in the simplest form. Choose the line of centres as x-axis and the common radical axis as y-axis. Then, as in Theorem III, § 8.21 the equation to any member of the system its

$$x^2+y^2+2gx+c=0 \qquad \ldots(1)$$

where c is fixed and g is variable.

Corresponding to different values of g we shall get different members of the coaxal system. It is easy to see that the common radical axis of any two members of the system represented by equation (1) is the line $x=0$ *i.e.*, the axis of y.

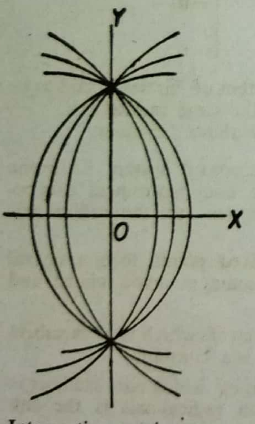
If $g=\pm\sqrt{c}$, the radius g^2-c becomes zero and the circles become point circles.

The points $(\pm\sqrt{c}, 0)$ which are the centres of the point circles belonging to the coaxal system (1) are called the *limiting points* of this system.

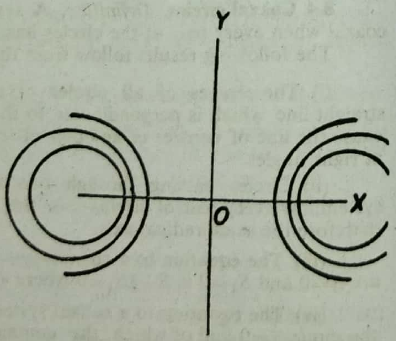
8.42 Intersection of members of coaxal system. Let two members of the coaxal system represented by the equation (1) of the preceding article be

$$x^2+y^2+2g_1x+c=0,$$
$$x^2+y^2+2g_2x+c=0.$$

The points of intersection of the circles are $(0, \pm\sqrt{-c})$. If, therefore, c is positive, no two members of the system have real points of intersection. The limiting points $(\pm\sqrt{c}, 0)$ of the system



Intersecting species



Non-intersecting species

are then real. If c is negative, the limiting points are imaginary, but any two members of the systems intersect in real points which are the two points in which any member meets the radical axis of the system.

The circles of coaxal system thus intersect in real or imaginary points according as the limiting points of the system are imaginary or real. In the former case, the circles are said to be of the **Intersecting Species.**

Examples

1. *Prove that the chords of intersection of a given circle and a given coaxal system of circles pass through a fixed point.* (*Lucknow*, 1961)

Solution. Let the given coaxal system of circles be $x^2+y^2+2gx+c=0$, and let the given circle be $x^2+y^2+2Gx+2Fy+C=0$. Then the chord of intersection is $2Gx+2Fy+C-c-2gx=0$.

Here g is variable. Hence this chord passes through the point of intersection of the lines $x=0$ and $2Gx+2Fy+C-c=0$, which is a fixed point.

2. *Find the coordinates of the limiting points of the coaxal system to which the circles*

$$x^2+y^2+4x+2y+5=0 \text{ and } x^2+y^2+2x+4y+7=0 \textit{ belong.}$$

(*Punjab*, 1964; *Magadh*, 1968; *Ranchi*, 1967)

Solution. The equation of the coaxal system is

$$x^2+y^2+4x+2y+5+\lambda\,(x^2+y^2+2x+4y+7)=0,$$

or
$$x^2+y^2+\frac{2\,(2+\lambda)\,x}{1+\lambda}+\frac{2\,(1+2\lambda)y}{1+\lambda}+\frac{5+7\lambda}{1+\lambda}=0.$$

The square of the radius of this circle is

$$\frac{(2+\lambda)^2+(1+2\lambda)^2-(5+7\lambda)\,(1+\lambda)}{(1+\lambda)^2}$$

and the coordinates of the centre are

$$\left(-\frac{2+\lambda}{1-\lambda'}-\frac{1+2\lambda}{1+\lambda}\right).$$

Equating the radius to zero, the equation in λ is

$$2\lambda^2+4\lambda=0.$$

i.e.,
$$\lambda=0 \text{ or } \lambda=-2.$$

Hence the limiting points are $(-2, -1)$ and $(0, -3)$.

3. Find the coordinates of the limiting points of the coaxal systems determined by the circles

$$x^2+y^2+2x-6y=0 \text{ and } 2y^2+2x^2-10y+5=0.$$

Ans. (1, 2), (3, 1).

4. Find the equation to the circle which belongs to the coaxal system of which the limiting points are $(1, -1)$, $2, 0)$ and which passes through the origin.

(*Andhra*, 1962)

Ans. $x^2+y^2+4y=0$.

5. Prove that the limiting points of a system of coaxal circles are inverse points with regard to every circle of the system.

6. Show that the polar of one limiting point of a coaxal system with respect to any circle of the system passes through the other limiting point. (*Lucknow*, 1960)

Solution. Let the coaxal system be

$$x^2+y^2+2gx+c=0. \quad ...(1)$$

where g is arbitrary and c fixed.

The limiting points are $(\sqrt{c}, 0)$, $(-\sqrt{c}, 0)$.

The polar of $(\sqrt{c}, 0)$ with respect to (1) is

$$x\sqrt{c}+g\,(x+\sqrt{c})+c=0.$$

i.e. $$(g+\sqrt{c})\,(x+\sqrt{c},=0.$$

Since $g\neq-\sqrt{c}$, the polar is the line $x+\sqrt{c}=0$, which passes through $(\sqrt{c}, 0)$, the other limiting point.

It can similarly be shown that the polar of $(-\sqrt{c}, 0)$ with respect to (1) passes through $(\sqrt{c}, 0)$.

Note. The above result is equivalent to the statement that *the limiting points of a system of coaxal circles are conjugate points with respect to every circle of the system.*

8.5 The orthogonal system. Let us find the equation to a circle which cuts orthogonally every member of a given system of coaxal circles.

Taking the line of centres as x-axis and the common radical axis as y-axis the equation to any circle of the given coaxal system is

$$x^2+y^2+2gx+c=0. \quad ...(1)$$

Let the circle whose equation is

$$x^2+y^2+2Gx+2Fy+k=0 \quad ...(2)$$

cuts the circle (1) orthogonally.

Then, from the condition of orthogonal intersection of two circles, we have

$$2gG=c+k. \quad ...(3)$$

Relation (3) being true for all values of g, we have

$$G=0,\ k=-c.$$

Hence, the equation of the circle which cuts every member of the given coaxal system orthogonally, is

$$x^2+y^2+2Fy-c=0. \quad ...(4)$$

where F is arbitrary.

Equation (4) represents a system of circles cutting the given coaxal system orthogonally. The orthogonal system represented by equation (4) is also *coaxal*, for the radical axis of any two circles of the system is the axis of x and the line of centres the axis of y.

Equation (4) is satisfied by the coordinates $(\pm\sqrt{c}, 0)$ of the given coaxal system (1). *Every member of the orthogonal system therefore passes through the limiting points of the given coaxal system.*

The limiting coordinate points of the orthogonal system are $(0, \pm\sqrt{-c})$, which satisfy equation (1). *The given coaxal system thus passes through the limiting points of the orthogonal system.* Further, if the limiting points of one system ar real, those of the other are imaginary.

Examples

1. *Show that, as λ varies, the circles*

$$x^2+y^2+2ax+2by+2\lambda\,(ax-by)=0$$

form a coaxal system. Find the equation of the radical axis.

Find also the equation of the circles which are orthogonal to all the circles of the above system.

(*Lucknow*, 1967)

Solution. Let the equation of the circle cutting the given system orthogonally be

$$x^2+y^2+2gx+2fy+c=0. \quad ...(1)$$

Then the relation

$$2a\,(1+\lambda)\,g+2b\,(1-\lambda)\,f=c$$

is true for all values of λ. Therefore in the above relation, the coefficient of λ and the constant term must separately vanish.

We then have

$$ag-bf=0,$$

and $$2ag+2bf=c.$$

From these $g=\dfrac{c}{4a}$, $f=\dfrac{c}{4b}$.

Substituting in equation (1), the orthogonal system is

$$x^2+y^2+\frac{c}{2}\left(\frac{x}{a}+\frac{y}{b}+2\right)=0.$$

2. Show that there are only two points each of which has the same polar with respect to every circle of a coaxal system and that they are the limiting points of the system. (*Lucknow*, 1967)

3. A circle cuts orthogonally two fixed non-intersecting circles. Prove that it passes through two fixed points on their line of centres.

Prove also that the two points are inverse points with regard to either of the given circles.

4. *If origin be a limiting point of a coaxal system, one of whose members is*

$$x^2+y^2+2\alpha x-2\beta y+c=0,$$

find the other limiting point and show that the orthogonal system is

$$(\alpha+\mu\beta)(x^2+y^2)-c(x+\mu y)=0.$$

(*Ranchi*, 1968)

Solution. The equation to the given coaxal system is obviously

$$x^2+y^2-2\alpha x-2\beta y+c+\lambda(x^2+y^2)=0,$$

i.e. $$x^2+y^2-\frac{2\alpha}{1+\lambda}x-\frac{2\beta}{1+\lambda}y+\frac{c}{1+\lambda}=0. \quad ...(1)$$

The centre of this circle is

$$\left(\frac{\alpha}{1+\lambda}, \frac{\beta}{1+\lambda}\right).$$

and the radius is

$$\frac{\{\alpha^2+\beta^2-c(1+\lambda)\}^{1/2}}{1+\lambda}.$$

The radius vanishes if

$$1+\lambda=\frac{\alpha^2+\beta^2}{c}.$$

The other limiting point is thus

$$\left(\frac{c\alpha}{\alpha^2+\beta^2}, \frac{c\beta}{\alpha^2+\beta^2}\right).$$

[It should be noted that we get only one finite value of λ. Actually, we always get a quadratic in λ so that the other value of λ in this case is infinite giving the origin as one limiting point.]

Let the orthogonal system be given by

$$x^2+y^2+2gx+2fy+k=0. \quad ...(2)$$

From the condition of orthogonal intersection of (1) and (2), we have

$$-\frac{2g\alpha}{1+\lambda}-\frac{2f\beta}{1+\lambda}=k+\frac{c}{1+\lambda},$$

or $$-(2g\alpha+2f\beta)=c+k(1+\lambda).$$

The above relation being true for all values of γ, we have

$$k=0, \text{ and } f=-\frac{c+2g\alpha}{2\beta}.$$

Hence equation (2) can be written as

$$x^2+y^2+2g\alpha-\frac{c+2g\alpha}{\beta}y=0,$$

or $$x^2+y^2-2g\left(\frac{c+2g\alpha}{2g\beta}y-x\right)=0.$$

Writing

$$\frac{c+2g\alpha}{2g\beta}=-\mu,$$

we get $$2g=-\frac{c}{\alpha+\mu\beta}.$$

The required orthogonal system is, therefore,

$$(x^2+y^2)(\alpha+\mu\beta)-c(x+\mu y)=0.$$

5. *Prove that the limiting points of the system*

$$x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+k)=0$$

subtend a right angle at the origin if

$$\frac{c}{g^2}+\frac{k}{f^2}=2.$$

Solution. The equation of the system is

$$x^2+y^2+\frac{2gx}{1+\lambda}+\frac{2fy\lambda}{1+\lambda}+\frac{c+k\lambda}{1+\lambda}=0.$$

The centre is $\left(\frac{-g}{1+\lambda}, \frac{-fy}{1+\lambda}\right)$ ...(1)

and radius is$=\sqrt{\frac{g^2}{(1+\lambda)^2}+\frac{f^2\lambda^2}{(1+\lambda)^2}-\frac{c+k\lambda}{1+\lambda}}$ ...(2)

For limiting points, radius should be zero. Thus

$$\lambda^2(f^2-k)-\lambda(c+k)+(g^2-c)=0. \quad ...(3)$$

Let λ_1 and λ_2 be the roots of this equation. Then

$$\lambda_1+\lambda_2=\frac{c+k}{f^2-k} \text{ and } \lambda_1\lambda_2=\frac{g^2-c}{f^2-k}.$$

The coordinates of limiting points L_1 and L_2 are

$$L_1:\left(\frac{-g}{1+\lambda_1},\ \frac{-f\lambda_1}{1+\lambda_1}\right) \text{ and } L_2:\left(\frac{-g}{1+\lambda_2},\ \frac{-f\lambda_2}{1+\lambda_2}\right).$$

If L_1L_2 subtends a right angle at the origin, then

$$(\text{gradiant of } OL_1)\times(\text{gradiant of } OL_2)=-1.$$

or $$\left(\frac{f\lambda_1}{g}\right)\left(\frac{f\lambda_2}{g}\right)=-1 \text{ or } f^2\lambda_1\lambda_2+g^2=0.$$

Putting the value of $\lambda_1\lambda_2$ the required condition is

$$f^2\left(\frac{g^2-c}{f^2-k}\right)+g^2=0$$

or $$2f^2g^2=cf^2+kg^2$$

giving $$\frac{c}{g^2}+\frac{k}{f^2}=2.$$

Examples on Chapter VIII

1. Find the equation of the line joining the points of intersection of the circles $x^2+y^2=4$ and $x^2+y^2-2x-4y+1=0$ and the length of the common chord.

Ans. $2x+4y=5$; $\sqrt{11}$.

2. Find the radical centre of the circles

$$x^2+y^2+4x+7=0, \quad x^2+y^2+y=0$$

and $$2x^2+2y^2+3x+5y+9=0.$$

Ans $(-2, -1)$.

3. Find the equation to the circles which intersect the circles

$$x^2+y^2-6y+1=0 \text{ and } x^2+y^2+4y+1=0.$$

orthogonally and touch the line $3x+4y+5=0$.

Ans. $x^2+y^2=1$; $4(x^2+y^2-1)-15x=0$.

4. AB is a diameter of a circle. Show that the polar of A with respect to the circle which cuts the first circle orthogonally passes through B.

5. Prove that the polar lines of a fixed point P with respect to the circles of a given coaxal system pass through a fixed point Q.

6. Two points P and Q are conjugate with respect to a given circle. Show that the circle on PQ as diameter is orthogonal to the given circle.

7. Show that the system of circles given by

$$x^2+y^2+2\lambda x+2\mu y+\nu=0$$

has a common orthogonal circle if λ, μ, ν satisfy an equation of the form $A\lambda+B\mu+C\nu+D=0$.

8. Find the equation to the circle cutting orthogonally the circles

$$x^2+y^2-2x+3y-7=0, \quad x^2+y^2+5x-5y+9=0,$$

and $$x^2+y^2+7x-9y+29=0. \qquad \textit{(Jodhpur, 1965)}$$

Ans. $x^2+y^2-16x-18y=4$.

9. Show that the locus of points, the tangents from which two given circles bear a constant ratio, is a coaxal circle.

10. A certain point has the same polar with respect to each of two circles; prove that a common tangent subtends a right angle at this point.

Hint. The point is a limiting point.

11. If a circle cuts orthogonally three circles $S_1=0$, $S_2=0$, $S_3=0$, prove that it cuts orthogonally any circle

$$\lambda_1S_1+\lambda_2S_2+\lambda_3S_3=0.$$

12. Show that if $S_1=0$, $S_2=0$, $S_3=0$ represent three circles of which each two cut orthogonally, the equation

$$l_1S_1+l_2S_2+l_3S_3=0$$

represents a real circle except in certain cases where it represents a straight line.

13. Find the limiting points of the coaxal system of circles of which the two members are $x^2+y^2-6x-6y+4=0$ and

$$x^2+y^2-2x-4y+3=0. \qquad \textit{(Lucknow, 1963)}$$

Ans. $(-1, 1)$, $(1/5, 3/5)$.

14. Find the limiting points of coaxal system of circles of which two members are $x^2+y^2-6x+12y+5=0$ and

$$3(x^2+y^2)+10x-20y+15=0. \qquad \textit{(Andhra, 1963)}$$

Ans. $(1, -2)$, $(-1, 2)$.

15. Find the equation of the circle which passes through the origin and belongs to the coaxal system of which the limiting points are $(1, 2)$ and $(4, 3)$. *(Lucknow, 1961; Vikram, 1963)*

Ans. $2(x^2+y^2)-x-7y=0$.

16. The point $(2, 1)$ is limiting point of a coaxal system of circle of which $x^2+y^2-6x-4y+3=0$ is a member. Find the equation of the radical axis and coordinates of the other limiting point.

(Lucknow, 1978)

Ans. $x+y+4=0$, $(-5, -6)$,

17. Find the radical axis of the circles $x^2+y^2+2g_1x+2f_1y+c_1=0$ and $x^2+y^2+2g_2x+2f_2y+c_2=0$, and show that the first of these circles will bisect the circumference of other, if

$$2g_2(g_1-g_2)+2f_2(f_1-f_2)=c_1-c_2.$$

18. Prove that the equation to two given circles can always be put in the form $x^2+y^2+ax+b=0$ and $x^2+y^2+a'x+b=0$, and that one of these circles will be within the other if aa' and b are both positive. (*U. P. C. S.*, 1967)

19. The circle $x^2+y^2+4x-6y+3=0$ is one of the circles of a coaxal system having as radical axis the line $2x-4y+1=0$. Find the circle of the system which touches the line $x+3y-2=0$.

(*I. A. S.*, 1968)

Ans. $x^2+y^2+2x-2y+2=0$, $x^2+y^2-2x+6y=0$.

20. The equation to a circle of a given coaxal system is

$$x^2+y^2+2gx+2fy+c=0,$$

and origin is a limiting point of this system. Prove that the equation to the orthogonal system is

$$(g+\mu f)(x^2+y^2)+c(x+\mu y)=0.$$

(*Gorakhpur*, 1964; *Lucknow*, 1966)

21. If A, B, C be the centres of three coaxal circles and t_1, t_2, t_3 be the lengths of tangents to them from any point, show that

$$BC.t_1^2+CA.t_2^2+AB.t_3^2=0. \qquad \text{(Vikram, 1966)}$$

22. If the conics $ax^2+2hxy+by^2+2gx+2fy+c=0$

and $\qquad a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$

intersect in four concyclic points show that

$$\frac{a-b}{h}=\frac{a'-b'}{h'} \qquad \text{(Agra, 1966)}$$

23. Show that the general equation of all circles cutting at right angles the circles represented by

$$x^2+y^2-2a_1x-2b_1y+c_1=0,\ x^2+y^2-2a_2x-2b_2y+c_2=0$$

is given by

$$\begin{vmatrix} x^2+y^2 & x & y \\ c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \end{vmatrix}+\lambda\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix}=0$$

Hint. The radical axis ($L=0$) of orthogonal system is the line of centres of the given circles. Find the equation to one circle ($S=0$) of the orthogonal system and which passes through the origin. The required equation is $S+\lambda L=0$.

24. Find the limiting points of the coaxal system of circles

$$x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+c')=0$$

and show that the square of the distance between them is

$$\frac{(c-c')^2+4(cf^2+c'g^2-f^2g^2)}{f^2+g^2}$$

25. If two circles cut orthogonally, prove that an indefinite number of pair of points can be found on their common diameter, such that either point has the same polar with respect to one circle that the other has with respect to the other. Also show that the distance between any such pair of points, subtends a right angle at one of the points of intersection of two circles.

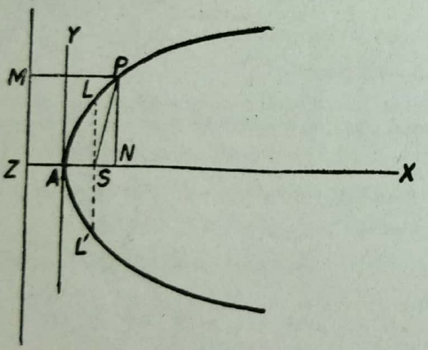
CHAPTER IX

THE PARABOLA

9.1 Equation of a parabola. We have seen in Chapter VI that the parabola is a conic section and that it is the locus of a point which moves such that its distance from a fixed point is equal to its distance from a fixed straight line. The fixed point is known as the **focus** and the fixed straight line **directrix** of the parabola.

We shall now use the above definition to find the equation of a parabola in standard form.

Let S be the focus, ZM the directrix ahd P a moving point on the parabola. Draw SZ perpendicular from S on the directrix. The middle point A of SZ will lie on the parabola, for $AS=AZ$. We call A the **vertex** of the parabola.



Let us choose the coordinate axis such that the origin lies at A, the x-axis along AS and the y-axis along the perpendicular to AS at A. Let the coordinates of P in any position be (x, y) and let AS be equal to a. The coordinates of S are then $(a, 0)$.

Also, PM, the perpendicular from P on the directrix, is equal to NZ where PN is the perpendicular from P on AS.

Now,

$$NZ=AN+AZ=AN+AS=x+a.$$

By definition,

$$PS=PM$$

i.e.,
$$(x-a)^2+y^2=(x+a)^2,$$

or
$$y^2=4ax,$$

which is the *standard form* of the equation to a parabola.

The line AS produced indefinitely is called the **axis** of the parabola, the perpendicular PN the **ordinate** of P and the double ordinate LSL' through the focus the **latus rectum** of the parabola.

From the equation of the parabola, we see that the y-coordinate of the points whose x-coordinate is a are $\pm 2a$ so that $SL=2a$ and SL' is also equal to $2a$. *The length of the latus rectum of the parabola is thus equal to* $4a$.

Since $AZ=a$, the *equation to the directrix* of the parabola is $x+a=0$.

9.11 Discussion of the equation to the parabola. From the equation $y^2=4ax$, where a is positive, we see that we get two equal and opposite values of y for any positive value of x. If x is negative, y becomes imaginary. The parabola therefore lies only on the positive side of the x-axis (which is also the axis of the parabola), and is *symmetrical* about it.

If $x=0$, the two values of y are each equal to zero. *The y-axis is, therefore, tangent to the parabola at the vertex.* If $x \to \infty$, so does y. The parabola therefore extends to infinity on either side of x-axis.

If can easily be seen that the equation

$$y^2=-4ax$$

represents an equal parabola lying wholly on the negative side of the x-axis.

The equations $x^2=4ay$ and $x^2=-4ay$ also represent equal parabolas but their axes lie along the y-axis. The parabola represented by the first equation lies on the positive side of y-axis while that represented by the second equation lies on the negative side of y-axis. The student should trace these parabolas as an exercise.

By a suitable transformation of the axes, namely, by transferring the origin to the vertex and rotating the coordinate axes such that the x-axis coincides with the axis of the parabola, the general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{...(1)}$$

when represents a parabola, will be reduced to the standard form

$$y^2=4Ax. \quad \text{...(2)}$$

By invariants, we have from (1) and (2),

$$ab-h^2=0,$$

which therefore is the *necessary condition* that the general equation (1) should represent a parabola.

The condition is *not* sufficient. The general equation (1) may also represent a pair of straight lines when $ab-h^2=0$. But if $ab-h^2=0$ and

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0,$$

it represents *only* a parabola.

The standard equation (2) expresses the fact that the square of the distance of any point on a parabola from the axis is equal to the product of the length of the latus rectum and the distance of the point from the tangent at the vertex. *This is an important characteristic of a parabola and is used in tracing it whenever the equation is given in the general form.*

As an illustration of the above, let us consider the parabola represented by the equation

$$(lx+my+n)^2=c\,(mx-ly+n').$$

Writing the equation as

$$\left(\frac{lx+my+n}{\sqrt{l^2+m^2}}\right)^2=\frac{c}{\sqrt{l^2+m^2}}\left(\frac{mx-ly+n'}{\sqrt{l^2+m^2}}\right)$$

we see that the equation expresses the fact that the square of the distance of the point (x, y) from the line $lx+my+n=0$ is $\dfrac{c}{\sqrt{l^2+m^2}}$ times its distance from the *perpendicular* straight line $mx+ly+n'=0$.

The line

$$lx+my+n=0 \qquad \ldots(3)$$

is, therefore, the axis and the line

$$mx-ly+n'=0 \qquad \ldots(4)$$

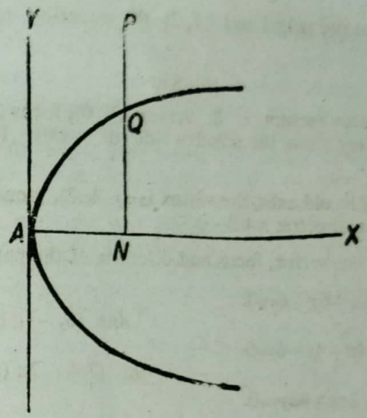
the tangent at the vertex of the given parabola. By solving (3) and (4) and simultaneous equations, we get the coordinates of the vertex. The latus rectum is the numerical value of expression

$$\frac{c}{\sqrt{l^2+m^2}}.$$

9.12 Position of a point relative to a parabola. We shall prove the following proposition :

The point (x', y') lies outside, on or inside the parabola $y^2=4ax$ according as the expression y'^2-4ax' is positive, zero or negative.

Let P be the point (x', y'). Draw PN perpendicular to x-axis. Let PN meet the parabola in Q. The abscissa of Q being x', $QN^2=4ax'$.



Now, P lies outside the parabola if $PN>QN$

i.e., if $PN^2>QN^2$,

i.e., if $y'^2>4ax'$.

Similarly, P lies inside the parabola if

$$y'^2<4ax'.$$

If $y'^2=4ax'$, P obviously lies on the parabola.

Examples

1. Find the equation of the parabola whose vertex is (0, 0) and focus (0, 1).

Ans. $x^2=4y$.

2. Find the equation of the parabola whose focus is $(-3, 0)$ and directrix $x+5=0$.

Ans. $y^2=4\,(x+4)$.

3. For the parabola $4(y-1)^2=-7(x-3)$, find (i) the latus rectum, and (ii) the coordinates of the focus and vertex.

Ans. $\frac{7}{4}$; $\left(\frac{41}{16}, 1\right)$; $(3, 1)$.

Hint. Transfer the origin to $(3, 1)$.

4. *Trace the parabola* $y^2-8x-4y-4=0$.

Solution. Writing the equation as

$$(y-2)^2=8(x+1)$$

and transferring the origin to $(-1, 2)$, the equation to the parabola becomes

$$y^2=8x,$$

of which the latus rectum is 8, vertex $(0, 0)$, focus $(2, 0)$ and the axis $y=0$ running along the positive side of x-axis. The directrix is $x+2=0$.

Referred to old axes, the vertex is $(-1, 2)$, focus $(1, 2)$ and the equation to the directrix $x+3=0$.

5. Find the vertex, focus and directrix of the parabolas

(i) $x^2-6x-6y+6=0$.

Ans. $(3, -\frac{1}{2})$; $(3, 1)$; $y+2=0$.

(ii) $y^2+4x+4y-3=0$.

Ans. $(7/4, -2)$; $(\frac{3}{4}, -2)$; $4x=11$.

(iii) $x^2-2ax+4ay=0$.

Ans. $\left(a, \frac{a}{4}\right)$; $\left(a, \frac{-3a}{4}\right)$; $4y=5a$.

6. A double ordinate of the parabola $y^2=4ax$ is of length $8a$; prove that the lines from the vertex to its two ends are at right angles.

7. If every member of a family of circles passes through a point and is tangential to a fixed line, then show that the locus of their centres is a parabola. (*Lucknow*, 1968)

8. *Chords of a parabola intersect the axis in a common point P which is on the same side of the vertex as the focus. If the distance of P from vertex be equal to the length of the latus rectum of the parabola, show that the chords subtend the same angle at the vertex.*

Solution. Let $y=mx+c$ be the equation to a chord of the parabola $y^2=4ax$. Then the equation to the pair of lines joining the vertex to the intersections of the chord with the parabola is

$$y^2=\frac{4ax(y-mx)}{c},$$

or
$$cy^2-4axy+4amx^2=0.$$

The angle between these lines is

$$\tan^{-1}\left(\frac{4\sqrt{a^2-amc}}{c+4am}\right).$$

The chord meets the axis in the point $\left(-\frac{c}{m}, 0\right)$.

Hence we have $-\frac{c}{m}=4a$, that is, $c+4am=0$.

Therefore the angle which any chord subtends at the vertex is $90°$.

9. A parabola is drawn to pass through A and B, the ends of a diameter of given circle of radius a, and to have as directrix a tangent to a concentric circle of radius b, the axes being AB and a perpendicular diameter; prove that the locus of the focus of the parabola is

$$\frac{x^2}{b^2}+\frac{y^2}{b^2-a^2}=1.$$

10. Show that the locus of the centre of the circle which touches the line $x+y=0$, and which passes through the point (a, a) lies on the parabola

$$x^2-2xy+y^2-4ax-4ay+4a^2=0.$$

Determine the coordinates of the vertex of the parabola.

Ans. $\left(\frac{a}{2}, \frac{a}{2}\right)$.

9.2 Tangent and other loci. By the methods used for the general equation in Chapter VI, or by regarding the equation $y^2=4ax$ as a particular case of the general equation, we have the following results :

The *tangent* at (x', y') on the parabola is

$$yy'=2a(x+x').$$

The *chords of contact* of tangent from (x', y') or the *polar* of (x', y') is

$$yy'=2a(x+x').$$

The equation of the *pair of tangents* from (x', y') is

$$(y^2-4ax)(y'^2-4ax')=\{yy'-2a(x+x')\}^2.$$

The equation of the *chord whose middle point is* (x', y') is

$$yy'-2ax=y'^2-2ax'.$$

9.3 Intersection of a straight line and a parabola. Let the equation of a straight line be ...(1)

$$y=mx+c$$

of a parabola

$$y^2=4ax. \qquad \ldots(2)$$

The x-coordinates of the points of intersection of (1) and (2) are the roots of the equation

$$(mx+c)^2=4ax,$$

i.e.,
$$m^2x^2+2\,(mc-2a)\,x+c^2=0.$$

This gives two values of x. If $m=0$, one root becomes infinite. We thus conclude that *every straight line cuts a parabola in two points. If the line is parallel to the axis of parabola, one of the point of intersection lies at infinity.*

If the line is a tangent to the parabola, the roots of the above quadratic should be coincident. The condition for this is

$$(mc-2a)^2=m^2c^2,$$

i.e.,
$$c=\frac{a}{m}.$$

The line $y=mx+\frac{a}{m}$ therefore is tangent to the parabola $y^2=4ax$ for all values of m.

If $y=mx+\frac{a}{m}$ passes through a fixed point (h, k), we have

$$k=mh+\frac{a}{m},$$

or
$$m^2h-mk+a=0.$$

This being a quadratic in m, two tangents can be drawn to the parabola from a given point (h, k). The tangents will be real and distinct only if $k^2>4ah$, *i.e.*, only if the point (h, k) lies outside the parabola.

Example. Find the coordinates of the point where the line $y=mx+\frac{a}{m}$ is tangent to the parabola $y^2=4ax$.

Let (x', y') be the point of contact.

Then
$$yy'=2a\,(x+x')$$

and
$$y=mx+\frac{a}{m}$$

represent the same straight line.

Comparing coefficients, we obtain

$$y'=\frac{2a}{m}=2mx'.$$

Hence,
$$x'=\frac{a}{m^2},\ y'=\frac{2a}{m}.$$

9.4 Parametric coordinates. It is sometimes convenient to express the coordinates of a point on the parabola $y^2=4ax$ in terms of a single variable rather than to have both variables x' and y'. If we use t as a single parameter, we immediately see that the point $(at^2, 2at)$ lies on the parabola. We thus say that the parametric equation of the parabola is

$$\mathbf{x=at^2,\ y=at.}$$

For the sake of brevity $(at^2, 2at)$ is usually referred to as the point 't'.

The parameter t is the contangent of the angle which the tangent at $(at^2, 2at)$ makes with the axis of x. The student should varify this as an exercise.

Examples

1. *Find the equation to the chord of the parabola $y^2=4ax$ joining the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$. Hence deduce the equation to the tangent at $(at^2, 2at)$.*

Solution. The required equation to the chord is

$$y-2at_1=\frac{2at_2-2at_1}{at_2^2-at_1^2}\,(x-at_1^2)$$

$$=\frac{2}{t_1+t_2}\,(x-at_1^2).$$

or
$$(t_1+t_2)\,y=2\,(x+at_1t_2).$$

If $t_2 \to t_1$, this becomes

$$t_1y=x+at_1^2.$$

Hence the equation of tangent at $(at^2, 2at)$ is

$$ty=x+at^2.$$

2. Show that the equation of the tangent to the parabola $y^2=4ax$ which makes an angle θ with its axis is

$$y=x\tan\theta+a\cot\theta.$$

3. If the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are the ends of a focal chord of the parabola $y^2=4ax$, then prove that $t_1t_2=-1$.

4. Prove that the straight line $lx+my+n=0$ will touch the parabola $y^2=4ax$ if $ln=am^2$.

5. Show that $y-x-2=0$ is a tangent to the parabola $y^2=8x$. What are the coordinates of the point of contact ?

Ans. (2, 4).

6. The inclinations θ and ϕ of two tangents to the parabola $y^2=4ax$ are given by

$$\tan\theta=\frac{1}{m},\ \tan\phi=-\frac{m}{2},$$

Show that, as m varies, the points of intersection of these tangents traces a line parallel to the directrix.

7. The angle between two tangents to the parabola $y^2=4ax$ is constant and equal to α. Prove that the locus of their point of intersection is given by

$$y^2-4ax=(a+x)^2\tan^2\alpha.$$

What happens to this locus if $\alpha=\frac{\pi}{2}$?

(*Punjab*, 1964; *Bhagalpur*, 1966)

8. Find the length of the side of an equilateral triangle inscribed in the parabola $y^2=4ax$ so that one angular points is at the vertex.

Ans. $8a\sqrt{3}$.

9. Show that the locus of the poles of chords of the parabola $y^2=4ax$ which subtend a constant angle θ at the vertex is the curve

$$4(y^2-4ax)=\tan^2\theta\,(x+4a)^2$$ (*Gorakhpur*, 1964)

10. Prove that the polar of $(-a, 2a)$ with respect to $x^2+y^2-2ax-3a^2=0$ touches the parabola $y^2=4ax$. (*Delhi*, 1956)

11. Chords of a parabola pass through a fixed point. Prove that the locus of their middle points is a parabola having its axis parallel to that of the given parabola.

(*Indian Audit and Accts. Service*, 1972)

12. Find the locus of the middle points of chords of a parabola which subtend a right angle at the vertex, and prove that all these chords pass through a fixed point on the axis of the curve.

(*Punjab*, 1965; *Lucknow*, 1960)

13. *Prove that if the difference of the squares of the perpendiculars on a moving line from two fixed points is constant, the line touches a fixed parabola.*

Solution. Taking the line joining the two points as the axis of x and the perpendicular bisector as the axis of y, the coordinates of the two points are $(a, 0)$ and $(-a, 0)$.

Let the moving line be $y=mx+c$.

Then we have

$$\frac{(am+c)^2-(-am+c)^2}{1+m^2}=\pm k.$$

Taking the plus sign (choice of sign does not affect the conclusion),

$$\frac{4amc}{1+m^2}=k,$$

i.e.,
$$c=\frac{k(1+m^2)}{4am}$$

$$c=\frac{k}{4am}+\frac{km}{4a}.$$

The equation to the moving line can now be written as

$$y=m\left(x+\frac{k}{4a}\right)+\frac{k}{4am}$$

which obviously touches the parabola

$$y^2=\frac{k}{a}\left(x+\frac{k}{4a}\right).$$

14. A tangent to the parabola $y^2+4bx=0$ meets the parabola $y^2=4ax$ at P and Q. Prove that the locus of the middle point of PQ is

$$y^2(2a+b)=4a^2x.$$

15. *Prove that the area of the triangle formed by the tangents from the point (x_1, y_1) to the parabola $y^2=4ax$ and the chord of contact is*

$$\frac{(y_1^2-4ax_1)^{3/2}}{2a}.$$ (*Gorakhpur*, 1967)

Solution. Let the tangents at $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meet in (x_1, y_1). Then, $x_1=at_1t_2$ and $y_1=a(t_1+t_2)$.

The length of the chord of contact is

$$\sqrt{(at_1^2-at_2^2)^2+(2at_1-2at_2)^2}$$
$$=a(t_1-t_2)\sqrt{(t_1+t_2)^2+4}$$
$$\frac{\sqrt{y_1^2-4ax_1}\cdot\sqrt{y_1^2+4a^2}}{a}.$$

The equation of the chord of contact is

$$yy_1-2a(x+x_1)=0.$$

The length of the perpendicular from (x_1, y_1) on this is

$$\frac{y_1^2-4ax_1}{\sqrt{y_1^2+4a^2}}.$$

Hence the required area is

$$\frac{(y_1^2-4ax_1)^{3/2}}{2a}.$$

9.5 The normal. The equation of the tangent at the point $(at^2, 2at)$ of the parabola $y^2=4ax$ is

$$ty=x+at^2.$$

Hence the equation of the normal at $(at^2, 2at)$, which is perpendicular to the tangent at this point is

$$t(x-at^2)+y-2at=0.$$

or

$$tx+y=2at+at^3.$$

If we replace $-t$ by m, the equation to a normal can be written as

$$\mathbf{y=mx-2am-am^3}.$$

The above line is obviously normal to a parabola at the point $(am^2, -2am)$.

9.51 Co-normal points. The equation of the normal at the point $(at^2, 2at)$ of the parabola $y^2=4ax$ is

$$tx+y=2at+at^3.$$

If this passes through a fixed point (h, k), then

$$th+k=2at+at^3,$$

i.e.,

$$at^3+t(2a-h)-k=0.$$

This is a cubic in t and has three roots. From the theory of equations, we know that the three roots of the above cubic at least one must be real.

Hence from a given point we can draw three normals to parabola of which at least one must be real.

If t_1, t_2, t_3 be the roots of the above cubic, we have

$$t_1+t_2+t_3=0.$$

Now, the coordinates y_1, y_2, y_3 of the feet of these normals are $2at_1, 2at_2, 2at_3$.

Hence,

$$y_1+y_2+y_3=0.$$

Thus, *the algebraic sum of the ordinates of the feet of the normals from any point to the parabola is zero.*

9.52 Circle through co-normal points. Let the normals at the points P, Q, R of the parabola $y^2=4ax$ meet in $(h; k)$. Let the coordinates of P, Q, R, be $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$; $(at_3^2, 2at_3)$.

Then, from the preceding article, t_1, t_2, t_3 are the roots of the equation

$$at^3+t(2a-h)-k)=0.$$

Putting $2at=y$, the ordinates of P, Q, R are the roots of the cubic

$$y^3+4a(2a-h)y-8a^2k=0. \quad ...(1)$$

Let the circle PQR be

$$x^2+y^2+2gx+2fy+c=0.$$

Eliminating x from the equations of the parabola and the above circle, we get

$$\frac{y^4}{16a^2}+y^2+2g\frac{y^2}{4a}+2fy+c=0,$$

i.e.,

$$y^4+y^2(16a^2+8ag)+32a^2fy+16a^2c=0. \quad ...(2)$$

Equation (2) gives the ordinates of the points of intersection of the parabola and the circle. Three roots of equation (2) are the same as the roots of equation (1). Let these idential roots be y_1, y_2, y_3 and let the fourth root be y_4.

Then, from (2),

$$y_1+y_2+y_3+y_4=0.$$

Also, from (1)

$$y_1+y_2+y_3=0.$$

From these, $\quad y_4=0.$

Hence *the circle through co-normal points passes through the vertex of the parabola.*

From the equation of the circle we now have

$$c=0.$$

Putting $c=0$ in (2) and ramoving the common factor y, we get

$$y^3+y(16a^2+8ag)+32a^2f=0. \quad ...(3)$$

Equations (1) and (3) being identical, we have, comparing coefficients,

$$4a(2a-h)=16a^2+8ag,$$

and

$$-8a^2k=32a^2f.$$

From these, $2g=-(h+2a)$, and $2f=-\frac{k}{2}$

The equation of the circle through the co-normal points P, Q, R is, therefore,

$$x^2+y^2-(h+2a)\,x-\tfrac{1}{2}ky=0.$$

Examples

1. The normal at the point $(at_1^2, 2at_1)$ on the parabola $y^2=4ax$ meets it again at $(at_2^2, 2at_2)$.

Prove that

$$t_2=-\left(t_1+\frac{2}{t_1}\right). \qquad \text{(Gorakhpur, 1968)}$$

Hint. Compare the coefficients in the equations

$$t_1x+y=2at_1+at_1^3 \text{ and } y-2at_1=\frac{2}{t_1+t_2}(x-at_1^2).$$

2. Prove that the chord of the parabola $y^2=4ax$ which is normal at the point whose abscissa is $2a$, subtends a right angle at the vertex.

3. Prove that the tangent drawn at one extremity of a focal chord of the parabola $y^2=4x$ is parallel to the normal drawn at the other extremity.

Hint. If t_1, t_2 are the extermities of a focal chord, $t_1t_2=-1$.

4. *Find the locus of the point of intersection of two normals to the parabola $y^2=4ax$ which are at right angles to one another.*

(*Delhi*, 1967; *Agra*, 1964; *Kashmir*, 1966)

Solution. The equations of the normals at the points t_1 and t_2, are

$$t_1x+y=2at_1+at_1^3,$$

and

$$t_2x+y=2at_2+at_2^3.$$

These intersect in the point (h, k), where

$$h=2a+a\,(t_1^2+t_2t_1+t_2^2) \qquad \ldots(1)$$

$$k=-at_1t_2\,(t_1+t_2). \qquad \ldots(2)$$

Also, since the normals are at right angles,

$$t_1t_2=-1. \qquad \ldots(3)$$

From (1) and (3), we get

$$h-2a=a\,\{(t_1+t_2)^2+1\}.$$

Using (2), this gives $h-3a=k^2/a$.

Hence the required locus is $y^2=a\,(x-3a)$.

Aliter. The equation of any normal to the parabola is

$$y=mx-2am-am^3.$$

Let it pass through (h, k). Then,

$$k=m\,(h-2a)-am^3,$$

or

$$am^3+m\,(2a-h)+k=0.$$

If m_1, m_2, m_3 be the roots of this equation, then

$$m_1+m_2+m_3=0. \qquad \ldots(1)$$

$$m_1m_2+m_2m_3+m_3m_1=\frac{2a-h}{a} \qquad \ldots(2)$$

$$m_1m_2m_3=-\frac{k}{a}. \qquad \ldots(3)$$

Let the normals of which the slopes are m_1 and m_2, be perpendicular. Then,

$$m_1m_2=-1. \qquad \ldots(4)$$

From (3) and (4), $m_3=\dfrac{k}{a}$. ...(5)

From (1), (2) and (4), we get

$$m_3^2=\frac{h-2a}{a}-1$$

$$=\frac{h-3a}{a}.$$

Substituting for m_3 from (5), we obtain

$$k^2=a\,(h-3a).$$

The locus of (h, k) is therefore

$$y^2=a\,(x-3a).$$

5. Prove that the locus of points such that two of three normals from them to the parabola $y^2=4ax$ coincide is

$$27ay^2=4\,(x-2a)^3. \qquad \text{(Punjab, 1963)}$$

Hint. Use *Aliter* of the preceding example and put $m_1=m_2$.

6. Show that the locus of the poles of normal chords of $y^2=4ax$ is the curve

$$(x+2a)\,y^2+4a^2=0.$$

(*U. P. C. S.*, 1967, *Ranchi*, 1968)

Hint. If (x', y') is the pole, the equations $yy'=2ax+2ax'$ and $y=mx-2am-am^3$ are identical. Compare coefficients and eliminate m.

7. *Prove that the shortest normal chord of the parabola $y^2=4ax$ is $6a\sqrt{3}$ and that its inclination to the axis is $\tan^{-1}\sqrt{2}$.* (*I. A. S.*, 1964)

Solution. If t_1, t_2 be the extremities of the normal at t_1 then, from Example 1

$$t_2=-\left(t_1+\frac{2}{t_1}\right).$$

If l be the length of the normal chord, then

$$l^2=(at_1^2-at_2^2)^2+(2at_1-2at_2)^2$$
$$=a^2(t_1-t_2)^2\{(t_1+t_2)^2+4\}$$
$$=16a^2\frac{(1+t_1^2)^3}{t_1^4}$$

For a maximum or minimum of l, $\frac{dl}{dt_1}=0$, from which, we get

$$t_1=\pm\sqrt{2}.$$

Substituting this value of t_1, we obtain

$$l^2=\frac{16a^2 3^3}{4},$$

i.e., $$l=6a\sqrt{3}.$$

Obviously this is the least value of l as the greatest value is $+\infty$.

The inclination of the normal to the axis is the numerical value of $\tan^{-1}(-t_1)$ *i.e.*, the inclination is $\tan^{-1}\sqrt{2}$.

8. Find the locus of middle points of the normal chords of the parabola $y^2=4ax$.

(*I. A. S.*, 1963; *Lucknow*, 1964; *Rorkee*; 1966)

Ans. $\frac{y^2}{2a}+\frac{4a^3}{y^2}=x-2a$.

9. The normals at two point P, Q on the parabola $y^2=4ax$ intersect on the curve. Show that the ordinates of P, Q are the roots of the quadratic

$$y^2=ky+8a^2=0,$$

where k is the ordinate of the point of intersection. Show also that PQ passes through a fixed point on the axis of the parabola.

(*Lucknow*, 1964; *Roorkee*, 1967)

Ans. $(-2a, 0)$.

10. The normals at the points Q, R on $y^2-4ax=0$ meet the parabola at the point P. Prove that the locus of the circumcentre of the triangle PQR is the parabola

$$2y^2-ax+a^2=0.$$

(*U. P. C. S.*, 1966)

Hint. Use the result of § 9.52. The coordinates of the circumcentre are $\left(a+\frac{h}{2}, \frac{k}{4}\right)$, where $k^2=4ah$.

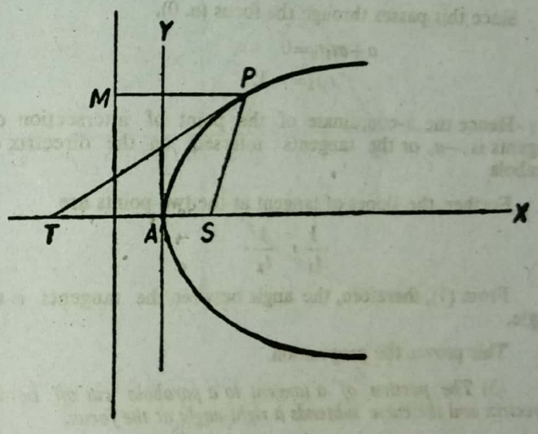
11. Show that the locus of points such that the sum of the angle which the three normals drawn from them to a parabola make with the axis of the parabola is constant, is a straight line.

9.6 Propositions on the parabola.

(1) *The tangent at any point of a parabola bisects the angle between the focal distance of the point and the perpendicular on the directrix from the point.*

Let P be any point $(at^2, 2at)$ on the parabola $y^2=4ax$. The equation of the tangent PT is

$$ty=x+at^2 \qquad ...(1)$$



Equation of the focal chord PS is

$$y=\frac{2t}{t^2-1}(x-a). \qquad ...(2)$$

Consequently, $\tan SPT=\dfrac{\dfrac{2t}{t^2-1}-\dfrac{1}{t}}{1+t\dfrac{2}{t^2-1}}=\dfrac{1}{t}.$

Now, $\frac{1}{t}$, being the slope of the tangent, is equal to

$$\tan PTS = \tan TPM.$$

i.e.,
$$\angle SPT = \angle MPT.$$

This proves the proposition.

Corollary. The normal at any point of a parabola bisects the angles between the focal chord and the diameter (*i.e.*, the line parallel to the axis) through that point.

(2) *The tangents at the extremeties of a focal chord of a parabola intersect at right angles on the directrix.*

(*Delhi*, 1962; *Lucknow*, 1966)

Let $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola $y^2=4ax$.

The coordinates of the point of intersection of tangents at these points are $\{at_1t_2, a(t_1+t_2)\}$.

The equation to the chord is

$$\tfrac{1}{2}(t_1+t_2)\, y = x + at_1t_2.$$

Since this passes through the focus $(a, 0)$,

$$a + at_1t_2 = 0$$

i.e.,
$$t_1t_2 = -1. \qquad \text{...(1)}$$

Hence the x-coordinate of the point of intersection of the tangents is $-a$, or the tangents intersect on the directrix of the parabola.

Further, the slopes of tangent at the two points are

$$\frac{1}{t_1}, \ \frac{1}{t_2}.$$

From (1), therefore, the angle between the tangents is a right angle.

This proves the proposition.

(3) *The portion of a tangent to a parabola cut off between the directrix and the curve subtends a right angle at the focus.*

Let P be a point $(at^2, 2at)$ on the parabola $y^2=4ax$. Then the tangents at P is

$$ty = x + at^2. \qquad \text{...(1)}$$

The equation of the directrix is

$$x + a = 0. \qquad \text{...(2)}$$

The coordinates of the point of intersection of (1) and (2) are

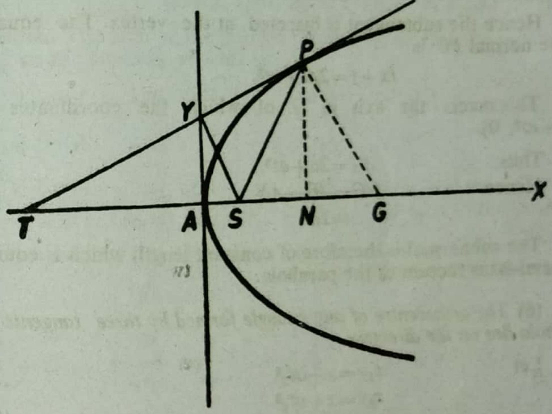
$$\left(-a, \frac{at^2-a}{t}\right).$$

The slope of the line joining this point to the focus $(a, 0)$ is $\frac{1-t^2}{2t}$, and the slope of the focal chord through P is $\frac{2t}{t^2-1}$.

The product of the two slopes is -1, which proves the proposition.

(4) *If SY be perpendicular to the tangent at a point P of a parabola, then Y lies on the tangent at the vertex and $SY^2 = AS.SP$.*

Let P be a point $(at^2, 2at)$ on the parabola $y=4ax$.



The equation to the tangent PT at P is

$$tx = x + at^2. \qquad \text{...(1)}$$

The perpendicular SY from $S\,(a, 0)$ on PT is

$$t(x-a)+y=0,$$

i.e.,
$$tx + y = at. \qquad \text{...(2)}$$

The x-coordinate of the point of intersection of (1) and (2) is evidently zero.

Hence, Y lies on the tangent at the vertex.

Further, $SY^2 = a^2(1+t^2)$
$= a(a+at^2)$
$= AS.SP.$

(5) *The subtangent of a point on a parabola is bisected at the vertex and the subnormal is of constant length.*

Definitions. Let the tangent and normal at a point P of a parabola meet the axis in T and G, respectively, and let PN be the ordinate of P; then NT is called *subtangent* and NG the *subnormal* of the point P.

Let P be the point $(at^2, 2at)$. The equhtion of the tangent PT is

$$ty = x + at^2.$$

This meets the axis in T of which the coordinates are $(-at^2, 0)$.

Thus, $AT = at^2 = AN.$

Hence the subtangent is bisected at the vertex. The equation of the normal PG is

$$tx + y = 2at + at^3.$$

This meets the axis in G of which the coordinates are $(2at + at^2, 0)$.

Thus $AG = 2a + at^2.$
Hence $NG = AG - AN$
$= 2a.$

The subnormal is therefore of constant length which is equal to the semi-latus rectum of the parabola.

(6) *The orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.*

Let $t_1 y = x + at_1^2$ ...(1)
$t_2 y = x + at_2^2$ ...(2)
and $t_3 y = x + at_3^2$ ...(3)

be three tangents to the parabola $y^2 = 4ax$ at the points 't_1', 't_2', 't_3'.

(1) and (2) intersect at the point

$$\{at_1t_2,\ a(t_1+t_2)\}.$$

The equation to the perpendicular from this point on (3) is

$$t_3 x + y = a(t_1 + t_2 + t_1t_2t_3). \qquad \text{...(4)}$$

Similarly, the equation of the perpendicular on (1) from the point of intersection of (2) and (3) is

$$t_1 x + y = a(t_2 + t_3 + t_1t_2t_3). \qquad \text{...(5)}$$

The x-coordinate of the orthocentre, which is the point of intersection of (4) and (5) is $-a$.

The orthocentre therefore lies on the directrix.

Examples

1. If from the vertex of a parabola a pair of chords be drawn at right angles to one another, and with these chords as adjacent sides a rectangle be constructed, prove that the locus of the outer corner of the rectangle is a parabola. (*Roorkee*, 1960)

2. A chord is a normal to a parabola and is inclined at an angle θ to the axis; prove that the area of the triangle formed by it and the tangents at its extremities is $4a^2 \sec^3\theta \operatorname{cosec}^3\theta$, where $4a$ is the latus rectum of the parabola.

3. *Prove that the area of a triangle inscribed in a parabola is twice the area of the triangle formed by the tangents at the vertices.*
(*Patna*, 1963; *Agra*, 1964; *U. P. C. S.*, 1977)

Solution. The area of the triangle formed by joining the points 't_1' 't_2' 't_3' on the parabola $y^2 = 4ax$ is

$$\frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix} = a^2 \begin{vmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{vmatrix}$$

$$= a^2\{t_1^2(t_2-t_3) + t_2^2(t_3-t_1) + t_3^2(t_1-t_2)\}$$

$$= a^2\{t_1t_2(t_1-t_2) + t_2t_3(t_2-t_3) + t_3t_1(t_3-t_1)\}$$

$$= -a^2 \begin{vmatrix} t_1t_2 & t_1+t_2 & 1 \\ t_2t_3 & t_2+t_3 & 1 \\ t_3t_1 & t_3+t_1 & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} at_1t_2 & a(t_1+t_2) & 1 \\ at_2t_3 & a(t_2+t_3) & 1 \\ at_3t_1 & a(t_3+t_1) & 1 \end{vmatrix}$$

= twice the area of the triangle formed by the tangents at 't_1', 't_2', 't_3',

4. *Prove that the circle described on any focal chord of a parabola as diameter touches the directrix.* (*I. A. S.*, 1968)

Solution. Let $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola $y^2 = 4ax$.

Then (cf. example 3 of § 9.4), $t_1t_2=-1$.

The equation of the circle described on this focal chord as diameter is

$$(x-at_1^2)(x-at_2^2)+(y-2at_1)(y-2at_2)=0,$$

i.e., $$x^2+y^2-ax(t_1^2+t_2^2)-2ay(t_1+t_2)-3a^2=0$$

where t_1t_2 has been put equal to -1.

To find where the circle meets the directrix we put $x=-a$. We then get

$$y^2-2ay(t_1+t_2)+a^2(t_1^2+t_2^2)-2a^2=0$$

or $$y^2-2ay(t_1+t_2)+a^2\{t_1^2+t_2^2+2at_1t_2\}=0,$$

or $$\{y-a(t_1+t_2)\}^2=0.$$

That is, the directrix meets the circle in two coincident points. This proves the proposition.

5. *Prove that the circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.*

Hint. The circle through 't_1', 't_2', 't_3' on the parabola $y^2=4ax$ is

$$x^2+y^2-ax(1+t_1t_2+t_2t_3+t_3t_1)-ay(t_1+t_2+t_3-t_1t_2t_3)$$
$$+a^2(t_1t_2+t_2t_3+t_3t_1)=0.$$

6. *A circle on any focal chord of a parabola as diameter cuts the curve again in P and Q. Show that PQ passes through a fixed point.* (*U. P. C. S.*, 1967)

Solution. As in example 4, the equation to the circle is

$$x^2+y^2-ax(t_1^2+t_2^2)-2ay(t_1+t_2)-3a^2=0.$$

Let $(at^2, 2at)$ be a point on this circle. Then

$$t^4+t^2\{4-(t_1^2+t_2^2)\}-4t(t_1+t_2)-3=0.$$

The roots of this equation are t_1, t_2, t_3, t_4 where $(at_3^2, 2at_3)$ $(at_4^2, 2at_4)$ are the coordinates of P and Q.

Then $$t_1+t_2+t_3+t_4=0,$$

and $$t_1t_2t_3t_4=-3$$

i.e., $$t_3t_4=3, \text{ since } t_1t_2=-1.$$

The equation to PQ is

$$(t_3+t_4)y=2(x+at_3t_4),$$

or $$y(t_1+t_2)+2x+6a=0.$$

This always passes through the fixed point $(-3a, 0)$.

9.7 Locus of middle points of parallel chords. Let (x', y') be the middle point of one of a system of chords of the parabola $y^2=4ax$ drawn parallel to

$$y=mx+c. \qquad \text{...(1)}$$

The equation to the chord is

$$yy'-2ax=y'^2-2ax'. \qquad \text{...(2)}$$

The slope of (2) is the same as that of (1).

Hence,

$$\frac{2a}{y'}=m.$$

The required locus, therefore, is

$$y=\frac{2a}{m}$$

which is a straight line parallel to the axis of the parabola.

Any line drawn parallel to the axis of a parabola is called a *diameter*. Thus a *diameter of a parabola bisects a system of parallel chords which are called the ordinates of that diameter.*

9.71 Tangent at the extremity of a diameter. The equation of the diameter of the parabola $y^2=4ax$ which bisects chords parallel to $y=mx+c$ is

$$y=\frac{2a}{m}.$$

This meets the parabola in the point whose coordinates are

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right).$$

The tangent at this point is $y=mx+a/m$ which is parallel to the given system of chords.

Hence, *the tangent at the extremity of a diameter of a parabola is parallel to the chord to which that diameter bisects.*

Note. This could be proved more easily by considering the fact that at the extremity of the diameter the chord which is bisected by that diameter becomes a tangent to the parabola.

9.72 Tangent at the ends of a chord of a parabola. The slope 'm' of the chord joining the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ of the parabola $y^2=4ax$ is $\dfrac{2}{t_1+t_2}$.

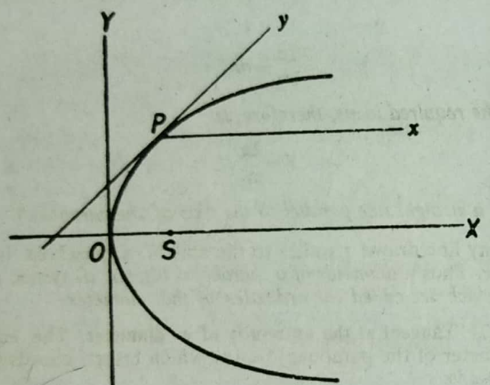
Hence the diameter bisecting this chord is

$$y=a(t_1+t_2).$$

But $a(t_1+t_2)$ is the y-coordinate of the point of intersection of the tangents at $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$.

Hence *the tangents at the extremities of any chord of a parabola meet on the diameter which bisects the chord.*

9.73 Equation of the parabola when the axes are a diameter and the tangent at the extremity of that diameter.



Let P be a point $(at^2, 2at)$ on the parabola $y^2=4ax$.

Transferring the origin to P and retaining the direction of the axes, equation of the parabola becomes

$$(y+2at)^2=4a(x+at^2)$$

Now let the new y-axis be rotated such that it coincides with the tangent at P. The new coordinates (X, Y) of any point on the parabola are connected with the coordinates (x, y) of the same point referred to rectangular axes through P by the equations (see Chapter IV).

$$x=X+Y\cos\omega,\ y=Y\sin\omega$$

where ω is the inclination of the tangent to the diameter.

The equation of the parabola referred to the tangent at P and the diameter through P is therefore

$$(Y\sin\omega+2at)^2=4a(X+Y\cos\omega+at^2),$$

that is $$Y^2\sin^2\omega+4aY(t\sin\omega-\cos\omega)=4aX.$$

Now, the slope of the tangent at P is $\tan\omega=\frac{1}{t}$.

Substituting this value of $\tan\omega$, the equation of the parabola can be written.

$$Y^2=4a(1+t^2)X.$$

But $$a(1+t^2)=PS=b,\text{ say.}$$

The equation of the parabola when the tangent at P and diameter through P are taken as coordinate axes is thus

$$y^2=4bx. \qquad ...(1)$$

It will thus be seen that *the equation $y^2=4bx$ is a particular case of the equation of the parabola referred to the tangent at any point and the corresponding diameter.*

From equation (1) we see that corresponding to any value of x there are two equal and opposite values of y. This, therefore, confirms the fact that chords parallel to a tangent are bisected by the diameter drawn through the point of contact of the tangent.

Examples

1. *Prove that the locus of the intersection of the normals at th ends of a system of parallel chords of a parabola is a straight line which is normal to the parabola.*

Solution. If y_1, y_2 be the ordinates of the extremities of a chord of the parabola $y^2=4ax$, drawn parallel to $y=mx+c$, $y_1+y_2=\frac{4a}{m}$.
If y_3 be the ordinate of a third point on the parabola such that the normal at these three points are concurrent, $y_3=-\frac{4a}{m}$, which fixes the third point. Hence the normals at the extremities of a system of parallel chords of the parabola intersect upon a fixed normal.

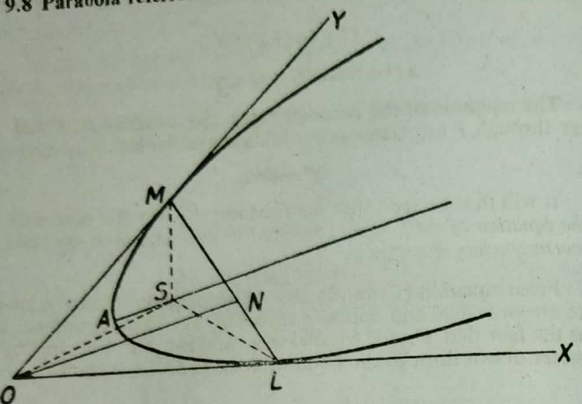
2. Prove that the locus of the middle points of the chords of a parabola passing through a fixed point is a parabola whose latus rectum is half that of the given parabola. (*Agra*, 1954)

3. P, Q and R are three points on a parabola and the chord PQ cuts the diameter through R in V. Ordinates PM and QN are drawn to this diameter. Prove that $RM.RN=RV^2$.

4. The ordinates of three points on a parabola are in geometrical progression. Show that the tangents at the first and third points meet on the ordinate of the second point.

5. If the diameter through any point O of a parabola meets any chord in P, and the tangents at the ends of that chord meet the diameter in Q, Q', show that $OP^2=OQ.OQ'$.

9.8 Parabola referred to two tangents.



We have seen above that the equation

$$y^2=4bx \qquad ...(1)$$

represents a parabola even when the axes are not rectangular.

If now we transform equation (1) to another set of oblique axes, we shall get an equation of the form (see Chapter IV)

$$(Ax+By)^2+2Gx+2Fy+C=0. \qquad ...(2)$$

Let us now find the equation to the parabola when the coordinate axes are two tangents to the parabola.

If OX and OY be the tangents and L and M the points of contact where $OL=a$ and $OM=b$, equation (2) will have two equal roots a, a when y is put equal to zero, and two equal roots b, b when x is put equal to zero.

Putting $y=0$,

$$A^2x^2+2Gx+C=0. \qquad ...(3)$$

Putting $x=0$,

$$B^2y^2+2Fy+C=0. \qquad ...(4)$$

From (3), $\quad A^2a=-G$ and $A^2a^2=C$.

From (4), $\quad B^2b=-F$ and $B^2b^2=C$.

From these, $\quad B=\pm A\frac{a}{b}$. $\qquad ...(5)$

Taking the negative sign, and substituting in equation (2)

$$\left(x-\frac{ay}{b}\right)^2-2ax-\frac{2a^2}{q}y+a^2=0,$$

or $$\left(\frac{x}{a}-\frac{y}{b}\right)^2 \; 2\left(\frac{x}{a}+\frac{y}{b}\right)+1=0.$$

or $$\left(\frac{x}{a}+\frac{y}{b}-1\right)^2=\frac{4xy}{ab},$$

or $$\frac{x}{a}+\frac{y}{b}-1=\pm 2\sqrt{\left(\frac{xy}{ab}\right)},$$

or $$\left(\sqrt{\frac{x}{b}}\pm\sqrt{\frac{y}{b}}\right)^2=1,$$

or, taking the plus sign,

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1 \qquad ...(6)$$

It is easy to see that if we take the plus sign in (5) and substitute in (2), we get two coincident straight lines

$$\left(\frac{x}{a}+\frac{y}{b}=-1\right)^2=0.$$

For any point lying on the parabola between L and M, $x<a$ and $y<b$. Therefore the radical signs in equation (6) are both positive.

For points lying beyond L, $x>a$ and $y<b$.

The equation representing the branch of the parabola beyond L is, therefore

$$\sqrt{\frac{x}{b}}-\sqrt{\frac{y}{b}}=1.$$

Similarly, the equation representing the branch of the parabola beyond M is

$$-\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1.$$

The two radical signs cannot both be negative.

9.81 Tracing of the parabola

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1.$$

We shall now determine the *focus, axis, vertex, directrix* and the *latus rectum* of the parabola

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{x}{b}}=1.$$

I. Focus. We shall make use of the following result which can easily be established by using rectangular axes.

If the tangents at L and M to a parabola meet in O, then (i) the angles OSL and OSM are equal, (ii) $OS^2=SL.SM$, S being the focus.

From these results it follows that the triangles OSL and OSM are similar. The angles SOL and SMO are, therefore, equal. The line OL then touches at O the circle through O, S and M.

The focus S thus lies on the circle through $(0, b)$ and touching the x-axis at the origin.

If the angle LOM be ω, the equation to the circle is

$$x^2+y^2+2xy \cos \omega=by.$$

The focus S, similarly, also lies on the circle through L and touching the y-axis at the origin.

The equation to this second circle is

$$x^2+y^2+2xy \cos \omega=ax.$$

The two circles intersect in the point

$$\left(\frac{ab^2}{a^2+2ab \cos \omega+b^2}, \frac{a^2b}{a^2+2ab \cos \omega+b^2}\right),$$

which is therefore the *focus* of the parabola.

II. Axis. If N be the middle point of LM, the line ON will be parallel to the axis AS of the parabola.

This follows from § 9.72. The coordinates of N are $\left(\frac{a}{2}, \frac{b}{2}\right)$, and, therefore, the equation to ON is $ay-bx=0$.

The *equation to the axis*, which is parallel to ON and passes through S is, therefore,

$$ay-bx=\frac{ab\,(a^2-b^2)}{a^2+2ab \cos \omega+b^2}.$$

III. Vertex. The vertex is the point where the axis meets the parabola. Solving for x and y, the equations to the axis and the parabola, we get

$$\left\{\frac{ab^2\,(b+a \cos \omega)^2}{(a^2+2ab \cos \omega+b^2)^2}, \frac{a^2b\,(a+b \cos \omega)^2}{(a^2+2ab \cos \omega+b^2)^2}\right\}$$

as the *coordinates of the vertex.*

IV. Directrix. We know that perpendicular tangents to a parabola meet on the directrix. If, therefore, we can find two points on OL and OM, the tangents from which to the parabola are respectively perpendicular to these lines, the line joining the two points will be the directrix of the parabola.

Let $(h, 0)$ be a point on OL. The equation to the line perpendicular to OP through this point is

$$y=m\,(x-h), \text{ where } 1+m \cot \omega=0.$$

The equation to the perpendicular line is thus

$$x+y \cos \omega=h. \qquad \ldots(1)$$

Writing the equation to the parabola as

$$\left(\frac{x}{a}-\frac{y}{b}\right)^2-2\left(\frac{x}{a}+\frac{y}{b}\right)+1=0,$$

the line (1) touches the parabola if the quadratic

$$\left(\frac{h-y \cos \omega}{a}-\frac{y}{b}\right)^2-2\left(\frac{h-y \cos \omega}{a}+\frac{y}{b}\right)+1=0.$$

has equal roots.

Writing the equation to the quadratic in descending powers of y, we see that the roots are equal if

$$\{bh\,(a+b \cos \omega)+ab\,(a-b \cos \omega)\}^2=(a+b \cos \omega)^2\,(b^2h^2-2ab^2h+a^2b^2)$$

i.e., If

$$h=\frac{ab \cos \omega}{a+b \cos \omega}.$$

The point on OL from which the tangent to the parabola is perpendicular to OL is

$$\left(\frac{ab \cos \omega}{a+b \cos \omega}, 0\right).$$

This point lies on the directrix.

Similarly, the point on OM which lies on the directrix is

$$\left(0, \frac{ab \cos \omega}{b+a \cos \omega}\right).$$

The *equation to the directrix* which joins the above two points is, therefore

$$x\,(a+b \cos \omega)+y\,(b+a \cos \omega)=ab \cos \omega.$$

V. Latus rectum. The length of the perpendicular from the focus $\left(\frac{ab^2}{a^2+2ab \cos \omega+b^2}, \frac{a^2b}{a^2+2ab \cos \omega+b^2}\right)$ on the directrix is (see Chapter IV)

$$\frac{2a^2b^2 \sin^2 \omega}{(a^2+2ab \cos \omega+b^2)^{3/2}}.$$

The latus rectum being double this length is

$$\frac{4a^2b^2 \sin \omega}{(a^2+2ab \cos \omega+b^2)^{3/2}}.$$

9.82 Equation of tangent. Let (x', y'), (x'', y'') be two points on the parabola

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1.$$

Then

$$\sqrt{\frac{x'}{a}}+\sqrt{\frac{y'}{b}}=1. \quad ...(1)$$

and

$$\sqrt{\frac{x''}{a}}+\sqrt{\frac{y''}{b}}=1. \quad ...(2)$$

The equation to the line joining the points (x', y') and (x'', y'') is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x'). \quad ...(3)$$

From (1) and (2),

$$\frac{\sqrt{x''}-\sqrt{x'}}{\sqrt{a}}=-\frac{\sqrt{y''}-\sqrt{y'}}{\sqrt{b}}. \quad ...(4)$$

Now, equation (3) can be written as

$$y-y'=\frac{(\sqrt{y''}+\sqrt{y'})(\sqrt{y''}-\sqrt{y'})}{(\sqrt{x''}+\sqrt{x'})(\sqrt{x''}-\sqrt{x'})}(x-x'),$$

which, with the help of (4), reduces to

$$y-y'=-\sqrt{\frac{b}{a}}\cdot\frac{\sqrt{y''}+\sqrt{y'}}{\sqrt{x''}+\sqrt{x'}}(x-x').$$

In the limit when $y''\to y'$, $x''\to x'$, the chord becomes tangent at (x', y'). *The equation of the tangent at (x', y') is, therefore*

$$y-y'=-\sqrt{\frac{b}{a}}\cdot\sqrt{\frac{y'}{x'}}(x-x'),$$

or

$$\frac{x}{\sqrt{ax'}}+\frac{y}{\sqrt{by'}}=\sqrt{\frac{x'}{a}}+\sqrt{\frac{y'}{b}}=1.$$

from relation (1).

Examples

1. If the lengths of tangents drawn from an external point to a parabola are a and b, and the angle between them be θ, show that the parameter (latus rectum) of the parabola is

$$\frac{4a^2b^2 \sin^2 \theta}{(a^2+4ab \cos \theta+b^2)^{3/2}}$$

2. A parabola, whose latus rectum is l, slides in contact with each of two fixed perpendicular straight lines. Find the locus of its vertex referred to these lines as axes. *(Lucknow, 1960)*

Ans. $16x^{2/3}y^{2/3}(x^{2/3}+y^{2/3})=l^2$.

Examples on Chapter IX

1. Find the equation of the parabola whose focus is $(3, -4)$ and directrix the line $6x-7y+5=0$. *(Rajasthan, 1964)*

Ans. $49x^2+84xy+36y^2-570x+750y+2100=0$.

2. Prove that the chord of the parabola $y^2=4ax$ whose equation $y-x\sqrt{2}+4a\sqrt{2}=0$ is a normal to the parabola and that its length is $6a\sqrt{3}$. *(Ranchi, 1968)*

3. Show that the directrix of a parabola is the polar of its focus. *(Punjab, 1966)*

4. Prove that the length of intercept on the normal at the point $P:(at^2, 2at)$ of the parabola $y^2=4ax$ made by the circle described on the line joining the focus and P as diameter is $a\sqrt{1+t^2}$. *(Agra, 1966)*

5. Show that the distance between a tangent to the parabola $y^2=4ax$ and the parallel normal is $a \sec^2\theta \operatorname{cosec}\theta$, where θ is the inclination of either of the axis of parabola. *(Jiwaji, 1967)*

Hint. The equation of a tangent is $y=mx+a/m$, $(m=\tan\theta)$ at a point $P:(a/m^2, 2a/m)$ of the parabola. The parallel normal is $y=mx-2am-am^3$. Find the length of perpendicular on the normal form $(a/m^2, 2a/m)$, and put $m=\tan\theta$.

6. A line PQ meets the parabola $y^2=4ax$ in R such that PQ is bisected at R. If the coordinates of P be (x_1, y_1), show that the locus of Q is the parabola

$$(y+y_1)^2=8a(x+x_1). \quad \textit{(I. A. S., 1967)}$$

7. Prove that the parabolas $y^2=ax$ and $x^2=by$, cut one another at an angle

$$\tan^{-1}\left[\frac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}\right].$$

8. Find the equation of the tangent common to $y^2=4ax$ and $x^2=4by$.

Ans. $y=mx+a/m$, where $m^3=(-b/a)$.

9. Any chord of the parabola and the perpendicular to the chord through the pole of the chord meet the axis of the parabola P and Q. Prove that P and Q are equidistant from the focus of the parabola.

10. From the point where any normal to the parabola $y^2=4ax$ meets the axis is drawn a line perpendicular to this normal. Prove that this normal always touches an equal parabola.

11. If t is the point $(at^2, 2at)$ on the parabola $y^2=4ax$, find the coordinates of the pole of the chord joining t_1 and t_2 with respect to the parabola. If the chord subtends a right angle at the focus of the parabola, show that the locus of the pole is the rectangular hyperbola

$$x^2-y^2+6ax+a^2=0. \qquad (Andhra,\ 1960)$$

12. If the tangents be drawn to the parabola $y^2=4ax$ from points on the line $x+4a=0$, show that the chords of contact will subtend a right angle at the vertex of the parabola.

(*Vikram*, 1965; *Patna*, 1969)

13. Find the locus of a point P when three normals drawn from it are such that one bisect the angle between the other two.

(*Gorakhpur*, 1962)

Ans. $27ay^2=(x-5a)^2\ (x-a)$.

14. The normal at any point P of the parabola $y^2=4ax$ meets the axis in G and the tangent at vertex in H. If A be the vertex and the rectangle $AGQH$ be completed, prove that the locus of Q is

$$x^3=a\ (2x^2+y^2). \qquad (Agra,\ 1965)$$

15. If the tangent to the parabola $y^2=4ax$ meets the axis in T and the tangent at the vertex A in Y, and the rectangle $TAYQ$ be completed, show that locus of Q is

$$y^2+ax=0. \qquad (Bihar,\ 1966)$$

16. Prove that the locus of the point of intersection of the normals at the ends of a system of parallel chords of a parabola is straight line, which is normal to the curve. (*U. P. C. S.*, 1968)

17. Find the locus of the point of intersection of normals at the ends of a focal chord of a parabola. (*Ranchi*, 1967)

Ans. $y^2=a\ (x-3a)$.

18. Show that the polar of any point on the circle $x^2+y^2-2ax-3a^2=0$ with respect to the circle $x^2+y^2+2ax-3a^2=0$ will touch the parabola $y^2+4ax=0$.

19. Two tangents OP and OQ are drawn to parabola represented in rectangular cartesian coordinates by $y^2=4x$, from the point O, the coordinates of which are (4, 5). Prove that the joining the mid-points of OP and OQ touches the parabola.

Hint. If P and Q be points 't_1' and 't_2' then $t_1t_2=4$, $t_1+t_2=5$, giving $t_1=4$, $t_2=1$.

20. PQ is a chord of a parabola normal at P; AQ is drawn from the vertex A; and through P a line is drawn parallel to AQ meeting the axis in R. Show that AR is double the focal distance to P.

21. O is the pole of a chord PQ of a parabola; prove that the perpendiculars from P, O, Q on any tangent to the curve, are in geometrical progression.

22. Prove that the locus of the middle points of all tangents drawn from points on the directrix to the parabola $y^2=4ax$ is $y^2\ (2x+a)=a\ (3x+a)^2$. (*Lucknow*, 1967)

23. Show that if two tangents to a parabola intercept a constant length on any fixed tangent, the locus of their point of intersection is another equal parabola.

24. Tangents are drawn at the ends of a normal chord of the parabola $y^2=4ax$. Show that the locus of their point of intersection is the curve

$$(x+2a)\ y^2+4a^3=0. \qquad (Lucknow,\ 1965)$$

25. Show that the locus of the point of intersection of tangents to $y^2=4ax$ which intercept a constant length d on the directrix is

$$(y^2-4ax)\ (x+a)^2=d^2x^2. \qquad (Andhra,\ 1961)$$

26. Show that the locus of the middle point of a variable chord of the parabola $y^2=4ax$ such that the focal distances of its extremities are in the ratio 2 : 1 is

$$9\ (y^2-2ax)^2=4a^2\ (2x-a)\ (4x+a)$$

27. Show that the locus of the middle point of chords of the parabola $y^2=4ax$ which are of constant length $2l$ is

$$(4ax-y^2)\ (y^2+4a^2)=4a^2l^2$$

28. The normals at P, Q, R on the parabola $y^2-4ax=0$ meet in a point on the line $y=k$. Prove that the sides of the triangle PQR touch the parabola $x^2-2ky=0$. (*Lucknow*, 1967)

29. If two normals to the parabola $y^2=4ax$ make complementary angles with the axis, show that their point of intersection lies on one of the curves

$$y^2=a\ (x-a),\ y^2=a\ (x-3a).$$

30. Tangents to the parabola $y^2=4ax$ are drawn at points whose abscissae are in the ratio $m^2:1$. Prove that the locus of their points of intersection is the curve

$$y^2=(m^{1/2}+m^{-1/2})^2\,ax. \qquad \text{(Osmania, 1960)}$$

31. The normals at the extremities of a chord of the parabola $y^2=4ax$ meet on the parabola. Show that the middle point of the chord lies on the parabola

$$y^2=2a\,(x+2a). \qquad \text{(Lucknow, 1964)}$$

32. If three normals from a point to the parabola $y^2=4ax$ cut the axis in points whose distances from the vertex are in A. P., show that the point lies on the curve

$$27ay^2=2\,(x-2a)^3.$$

(*Agra*, 1962; *Gorakhpur*, 1966)

33. Show that if a chord of the parabola $y^2=4ax$ touches the parabola $y^2=4bx$, the tangents at its extremities meet on the parabola $by^2=4a^2x$, and the normals on the curve

$$(4a-b)^3y^2=4b^2\,(x-2a)^3.$$

34. A variable chord of a given parabola passes through a fixed point. The circle on this chord as diameter cuts the parabola again at two other points. Prove that the lines joining these two other points passes through another fixed point of which the ordinate is equal in magnitude to the ordinate of the first point.

35. Prove that the area of the triangle formed by the normals to the parabola $y^2=4ax$ at the points 't_1', 't_2', 't_3' is

$$\tfrac{1}{2}a^2\,(t_2-t_3)\,(t_3-t_1)\,(t_1-t_2)\,(t_1+t_2+t_3)^2.$$

36. Tangents to the parabola $y^2=4ax$ at two points P and Q meet in (α, β). Show that PQ is equal to

$$\frac{\sqrt{(\beta^2-4a\alpha)\,(\beta^2+4a^2)}}{a}$$

and that the normals at P and Q intersect at the point

$$\left(2a-\alpha+\frac{\beta^2}{a},\ \frac{-\alpha\beta}{a}\right).$$

37. Prove that the length of any focal chord of a parabola is four times the distance of the focus from the point where the diameter bisecting the chord meets the parabola. (*I. A. S.*, 1967)

38. Through each point of the straight line $x=my+b$ is drawn a chord of the parabola $y^2=4ax$ which is bisected by that point. Prove that the chord touches the parabola

$$(y+2am)^2=8a\,(x-b).$$

39. Prove that the middle point of the intercept made on a tangent to a parabola by the tangents at two points P and Q lies on the tangent which is parallel to PQ.

Solution. Let the parabola be $y^2=4ax$ and the coordinates of P, Q and the point of contact of third tangent be $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at^2, 2at)$. Then the coordinates of the middle point of the intercept are

$$\left\{\tfrac{1}{2}\,a\,(t_1+t_2)\,t,\ a\left(t+\frac{t_1+t_2}{2}\right)\right\}$$

Equation of PQ is

$$y-2at_1=\frac{2}{t_1+t_2}\,(x-at_1^2).$$

Tangent parallel to PQ is

$$y=\frac{2x}{t_1+t_2}+\tfrac{1}{2}a\,(t_1+t_2).$$

The coordinates of middle point satisfy this equation.

40. Find the equation to two tangents which touch both $x^2+y^2=2a^2$ and $y^2=8ax$. Also find the coordinates of the four points of contact.

Ans. $y=\pm(x+2a)$; $(-a, \pm a)$, $(2a, \pm 4a)$.

41. Show that three normals can be drawn from a point P (h, k) to the parabola $y^2=4ax$. If Q, R, S be the feet of the normals, prove that :

(i) The centroid of the triangle QRS lies on the axis of parabola.

(ii) The circumcircle of triangle QRS passes through the vertex of the parabola. (*I. A. S.*, 1975)

42. Prove that the line $Ax+By+C=0$ will touch the parabola

$$(x-x')^2+(y-y')^2=\frac{(lx+my+n)^2}{l^2+m^2},$$

if

$$(A^2+B^2)\,(lx'+my'+n)=2\,(Al+Bm)\,(Ax'+By'+C).$$

(*Lucknow*, 1963)

Hint. The point (x', y') is the focus, $lx+my+n=0$ is the directrix of the parabola. Use proposition 1 of § 9.6.

43. The normals of the point Q, R on the parabola $y^2=4ax$ meet on the parabola at the point P. Show that the locus of the orthocentre of the triangle PQR is $y^2=a\,(x+6a)$.

Solution. Let the coordinates of P, Q, R be $(at_r^2, 2at_r)$, $r=1, 2, 3$. The equation of perpendiculars from P and R on the opposite sides are

$$y-2at_1=\left(\frac{t_2+t_3}{2}\right)(x-at_1^2)$$

and $$y-2at_3=-\left(\frac{t_1+t_3}{2}\right)(x-at_3^2).$$

If (α, β) be the point of intersection of these lines, then

$$\alpha=-4a-(t_1t_2+t_2t_3+t_3t_1)$$

and $$\beta=2a(t_1+t_2+t_3)+\frac{a}{2}(t_1+t_2)(t_2+t_3)(t_3+t_1).$$

Also t_1, t_2, t_3 are the roots of the equation

$$at^3+(2a-h)t-k=0.$$

Consequently $t_1+t_2+t_3=0$, $t_1t_2+t_2t_3+t_3t_1=\frac{2a-h}{a}$,

and $$t_1t_2t_3=\frac{k}{a}.$$

Therefore $\alpha=h-6a$ and $\beta=-\frac{1}{2}k$. But (h, k) is the same point as P which lies on the parabola. Therefore $k^2=4ah$. Eliminating (h, k), the locus of (α, β) is $y^2=a(x+6a)$.

44. A parabola touches two given straight lines. If its axis passes through the point (p, q), referred to in these lines as coordinate axes, prove that the focus lies on the curve

$$x^2-y^2-px+qy=0.$$

Solution. From § 9.8 II, the axis of the parabola passes through (p, q), therefore

$$aq-bp=\frac{ab(a^2-b^2)}{a^2+2ab\cos\omega+b^2}. \qquad \ldots(1)$$

Again from § 9.81, I, the focus (h, k) is given by

$$h=\frac{ab^2}{a^2+2ab\cos\omega+b^2},\ k=\frac{a^2b}{a^2+2ab\cos\omega+b^2}.$$

From these, we obtain

$$h^2-k^2=\frac{a^2b^4-a^4b^2}{(a^2+2ab\cos\omega+b^2)^2}$$

and $$qk-ph=\frac{a^2bq-ab^2p}{(a^2+2ab\cos\omega+b^2)^2}$$

Now $h^2-k^2+qk-ph=0$ on using (1). Hence the required locus is

$$x^2-y^2+qy-px=0.$$

———

CHAPTER X

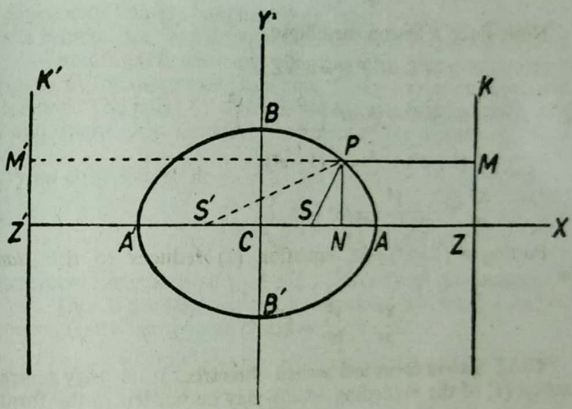
THE ELLIPSE

10.1 Definition. A conic section in which the eccentricity is less than unity, is called an ellipse.

Thus ellipse is the locus of a point which moves such that its distance from a fixed point (the focus) is e (<1) times its distance from a fixed straight line (the directrix).

e is called the eccentricity of the ellipse.

10.11 Equation of the ellipse



Let S be the focus, ZK the directrix and SZ the perpendicular from S to the directrix.

Let e be the eccentricity of the ellipse.

Since $e<1$, we shall have two points A and A' on ZS and $Z'S'$ produced such that $SA=e.AZ$ and $SA'=e.A'Z$.

By the definition of the ellipse, both A and A' lie on the ellipse.

Let $$AA'=2a.$$

Now, $$AA'=AS\pm SA$$
$$=e(AZ+A'Z)$$
$$=2e.CZ, \text{ where } C \text{ is the middle point of } AA',$$

Therefore, $CZ=\dfrac{a}{e}\cdot$

Further, $SA'-SA=e\,(A'Z-AZ)=e.AA'=2ae.$

But, $SA=CA-CS,\ SA'=CS+CA'=CS+CA.$

Therefore, $SA'-SA=2CS=2ae.$

i.e., $CS=ae.$

Let C be the origin, CA the x-axis and the line CB through C perpendicular to CA the y-axis.

Let P be any point (x, y) on the ellipse. Draw PM perpendicular to the directrix and PN perpendicular to CA.

The coordinates of the focus S are $(ae, 0)$ and the equation to the directrix ZK is $x=\dfrac{a}{e}\cdot$

Now, since P lies on the ellipse,

$$PS=e.PM=e.NZ,$$

i.e., $$(x-ae)^2+y^2=e^2\left(\frac{a}{e}-x\right)^2$$

or $$x^2\,(1-e^2)+y^2=a^2\,(1-e^2)$$

or $$\frac{x^2}{a^2}+\frac{y^2}{a^2\,(1-e^2)}=1. \qquad \text{...(1)}$$

Putting $a^2\,(1-e^2)=b^2$, equation (1) reduces to the *standard form*

$$\mathbf{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.}$$

10.12 Second focus and second directrix. It is easy to see that equation (1) of the preceding article may be written in the form

$$x^2+2aex+a^2e^2+y^2=e^2x^2+2aex+a^2$$

or $$(x+ae)^2+y^2=e^2\left(x+\frac{a}{e}\right)^2$$

If therefore S' be the point $(-ae,\ 0)$, and $Z'K'$ the line $x=-\dfrac{a}{e}$, the above equation expresses the fact that

$$S'P=e.NZ'$$
$$=e.PM',$$

where PM' is the perpendicular from P on $Z'K'$.

If, therefore, we take S' as the focus and $Z'K'$ as the directrix, the eccentricity having its previous value, we shall obtain the same curve.

Thus there exists a second focus and a second directrix for the ellipse.

10.13 Axes, foci and latus rectum. We have seen above that the equation to an ellipse in its standard form is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad \text{...(1)}$$

Putting $y=0$, $x=\pm a$. The ellipse that meets the x-axis in two points A, A' of which the coordinates are $(a, 0)$ and $(-a, 0)$.

A and A' are called the vertices of the ellipse, and AA' $(=2a)$ is called the *major axis.*

Putting $x=0$, $y=\pm b$. The ellipse thus cuts the y-axis in B, B' of which the coordinates are $(0, b)$ and $(0, -b)$. BB' $(=2b)$ is called the *minor axis* of the ellipse.

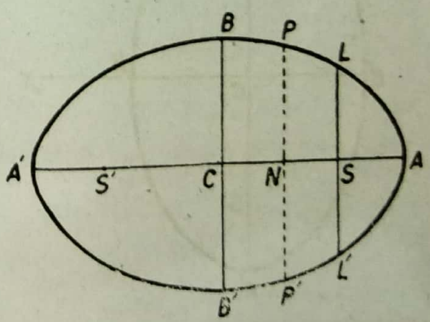
Since equation (1) contains only even powers of x and y, the ellipse is symmetrical with regard to both coordinates axes or its major and minor axes. Also, any chord of the ellipse through C is bisected at C, for the points (x, y) and $(-x, -y)$ simultaneously lie on the curve. The point C, which is the point of intersection of the major and minor axes, therefore the *centre* of the ellipse.

From equation (1),

$$x=\pm a\sqrt{1-\frac{y^2}{b^2}}\,.$$

x is therefore imaginary if $|\,y\,|>b$. Similarly y is imaginary if $|\,x\,|>a$. The ellipse thus lies in the region $-a\leqslant x\leqslant a,\ -b\leqslant y\leqslant b$. It is consequently limited and closed.

The points S, S' of which the coordinates are $(ae, 0)$ $(-ae, 0)$ are the *foci* of the ellipse.



The line PN perpendicular to the major axis from a point of the ellipse is called the *ordinate* of P. If PN produced meets the ellipse again in P', PNP' is called the *double ordinate* of P.

A double ordinate through a focus, as LSL', is called a latus rectum of the ellipse.

CS being equal to ae, we get from equation (1)

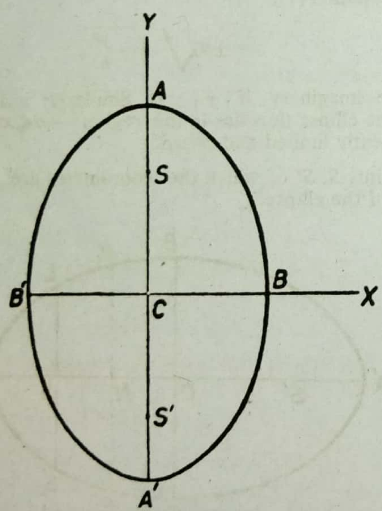
$$e^2+\frac{SL^2}{b^2}=1,$$

i.e., $$SL^2=b^2(1-e)^2=\frac{b^4}{a^2};$$

or $$SL=\frac{b^2}{a}.$$

The latus rectum of the ellipse is of length $\frac{2b^2}{a}$.

Note. If $b>a$, the major axis of the ellipse lies along the y-axis and is of length $2b$. The minor axis lies along the x-axis and is of length $2a$.



The coordinates of the focci are $(0, \pm be)$ and the eccentricity

is equal to $\sqrt{1-\frac{a^2}{b^2}}$. The equations of the directrix are

$$y=\pm\frac{b}{e}.$$

10.14 Positions of a point with regard to an ellipse.

The point (x', y') is inside, on or outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *according as the expression*

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1$$

is negative, zero or positive.

The proof of this proposition is similar to that for the parabola (§ 9.12) and is left as an exercise for the students.

10.15 A Geometrical property. If PN be the ordinate of $P:(x, y)$ on the ellipse (see figure of § 10.11), then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

i.e., $$\frac{CN^2}{a^2}+\frac{PN^2}{b^2}=1,$$

or $$\frac{PN^2}{BC^2}=1-\frac{CN^2}{CA^2}$$

$$=\frac{(CA+CN)(CA-CN)}{CA^2}$$

$$=\frac{NA'.AN}{CA^2}$$

Hence, $$\frac{PN^2}{AN.NA'}=\frac{BC^2}{AC^2}.$$

Similarly, if PN' be drawn perpendicular to the minor axis,

$$\frac{PN'^2}{BN'.N'B'}=\frac{AC^2}{BC^2}.$$

10.16 Sum of the focal distances of a point. Let P be a point (x, y) on the ellipse (see figure of § 10.11), then

$$PS=e.PM=e.NZ=e(CZ-CN)$$

$$=e\left(\frac{a}{e}-x\right)=a-ex,$$

and $$PS'=e.PM'=e(CZ'+CN)=a+ex.$$

Hence, $$PS+PS'=2a.$$

The sum of the focal distances of any point on the ellipse is therefore equal to the minor axis.

Note. The above property gives us a mechanical method of tracing an ellipse.

Fasten the ends of a piece of inextensible thread to two fixed points. Put a pencil point on the thread and turn it round such that the two portions of the thread between it and the fixed points are always tight. The curve traced will be an ellipse having its foci at the fixed points.

10.17 Equation referred to two perpendicular lines.

From the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we easily infer that if the perpendicular distance p, p' of a moving point $P:(x, y)$ from two perpendicular coplanar straight lines $u_1=0$, $u_2=0$ respectively satisfy the equation

$$\frac{p^2}{a^2}+\frac{p'^2}{b^2}=1,$$

the point P describes an ellipse in the plane of the given lines. The centre of the ellipse is the point of intersection of $u_1=0$, and $u_2=0$, and if $a>b$, the major axis lies along $u_2=0$, and the minor axis along $u_1=0$. The lengths of the major and minor axes in this case are respectively $2a$ and $2b$.

10.18 Limiting cases of an ellipse.

The equation to an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $$b^2=a^2(1-e^2).$$

We shall consider the following two cases :

Case I

Let $e \to 0$, a remaining finite. As $b \to a$, the foci approach the centre and the ellipse becomes almost a circle.

A circle is therefore the limiting case of an ellipse of which the eccentricity tends to zero.

Case II.

Transfer the origin to $(-a, 0)$. The equation to the ellipse referred to new axes becomes

$$\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1,$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2x}{a},$$

or $$y^2+\frac{b^2}{a^2}x^2=\frac{2b^2}{a}x. \quad ...(1)$$

Now, let a and b separately tend to infinity such that $\lim \frac{b^2}{a}=k$, a finite quantity.

Then, $$\lim \frac{b^2}{a^2}=\lim \frac{k}{a}=0,$$

and equation (1) reduces to

$$y^2=2kx,$$

which is the equation to a parabola.

A parabola is thus the limiting case of an ellipse of which the axes are of infinite length but the latus rectum is finite.

Note. It is easy to find two quantities a and b each tending to infinity such that $\lim \frac{b^2}{a}$ is a finite quantity. For example, if $a=n$, $b=\sqrt{n}$, a and b both tend to infinity as $n\to\infty$, but $\lim \frac{b^2}{a}=1$, a finite quantity.

Examples

1. *Find the eccentricity, the foci and the latus rectum of the ellipse*

$$2x^2+5y^2=20. \qquad \textit{(Aligarh, 1959)}$$

Solution. The equation to the ellipse can be written as $\frac{x^2}{10}+\frac{y^2}{4}=1$. The semi-major axis is $\sqrt{10}$, and the semi-minor axis is 2. The eccentricity is $\sqrt{1-\frac{4}{10}}=\sqrt{\frac{3}{5}}$. The coordinates of the foci are $(\pm\sqrt{6}, 0)$ and the latus rectum is $\frac{4\sqrt{10}}{5}$.

2. Find the eccentricity the foci and the latus rectum of the ellipse

$$5x^2+4y^2=2.$$

Ans. $\frac{1}{\sqrt{5}}, \left(0, \pm\frac{1}{\sqrt{10}}\right), \frac{4\sqrt{2}}{5}$.

Hint. The major axis lies along the y-axis.

3. Find the equation to the ellipse whose focus is the point $(-1, 1)$, whose directrix is the straight line $x-y+3=0$ and whose eccentricity is $\frac{1}{2}$. (*Calcutta*, 1952)

Ans. $7x^2+7y^2+2xy+10x-10y+7=0.$

4. Find the eccentricity and latus rectum of the ellipse
$$4x^2+9y^2-8x-36y+4=0.$$
Ans. $\dfrac{\sqrt{5}}{3}, \dfrac{8}{3}$.

Hint. The equation to the ellipse can be written as
$$\frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}=1.$$

5. *Find the centre and eccentricity of the ellipse*
$$4(x-2y+1)^2+9(2x+y+2)^2=5.$$

Solution. The given equation can be written as
$$4\left(\frac{x-2y+1}{\sqrt{5}}\right)^2+9\left(\frac{2x+y+2}{\sqrt{5}}\right)^2=1.$$

Putting $\dfrac{x-2y+1}{\sqrt{5}}=p,\ \dfrac{2x+y+2}{\sqrt{5}}=p'$,

the above equation becomes $\dfrac{p^2}{\frac{1}{4}}+\dfrac{p'^2}{\frac{1}{9}}=1.$

The semi-major axis is of lengths $\frac{1}{2}$ and the semi-minor axis is of length $\frac{1}{3}$. The equations to the major and minor axes are respectively $2x+y+2=0$ and $x-2y+1=0$. The coordinates of the centre are, therefore, $(-1, 0)$ and the eccentricity is
$$\sqrt{1-\frac{4}{9}}=\frac{\sqrt{5}}{3}$$

6. Find the equation to the ellipse whose axes are of lengths 6 and $2\sqrt{6}$ and their equations $x-3y+3=0$ and $3x+y-1=0$ respectively. *Ans.* $21x^2-6xy+29y^2+6x-58y-151=0.$

7. If the angle between the lines joining the foci of an ellipse to an extremity of the minor axis is 90°, find the eccentricity. Find also the equation of the ellipse if the major axis is $2\sqrt{2}$ units in length.

Ans. $\dfrac{1}{\sqrt{2}}$, $x^2+2y^2=2.$

8. Show that the triangle whose vertices are the points $(3, -1)$ $(1, 2)$ and $(-2, -1)$ lies inside the ellipse $2x^2+3y^2=30$.

9. Show that the sum of the reciprocals of the squares of any two diameters of an ellipse which are at right angles to one another is constant. (*U. P. C. S.*, 1977)

Hint. If $(r\cos\theta, r\sin\theta)$ be any point on the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \text{then}\ \frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}.$$

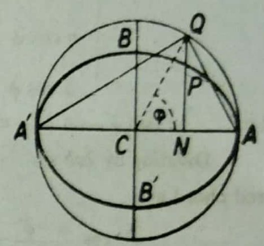
For the perpendicular diameter θ, is replaced by $\theta+\dfrac{\pi}{2}$.

10. A straight rod of given length slides between two fixed bars which include an angle of 90°. Show that the locus of a point on the rod which divides it in a given ratio is an ellipse. If this ratio be $\frac{1}{2}$, show that the eccentricity of the ellipse is $\dfrac{\sqrt{3}}{2}$.

10.2 Auxiliary circle.

Definition. The circle described on the major axis of an ellipse as diameter is called the auxiliary circle of the ellipse.

We shall prove that the distances from the major axis of any point on an ellipse and its corresponding point on the auxiliary circle are in the ratio of the minor and major axes of the ellipse.



Let P be a point on a ellipse whose major and minor axes are AGA' and BCB'. Let the ordinate PN, produced beyond P, meet the auxiliary circle in Q. Then P and Q are called *corresponding ponits.*

From § 10.15
$$\frac{PN^2}{AN.NA'}=\frac{BC^2}{AC^2}. \qquad ...(1)$$

Also from the right angled triangle AQA',
$$AN.NA'=QN^2. \qquad ...(2)$$

From (1) and (2),
$$\frac{PN^2}{QN^2}=\frac{BC^2}{AC^2},$$
or
$$\frac{\mathbf{PN}}{\mathbf{QN}}=\frac{\mathbf{b}}{\mathbf{a}}.$$

Corollary. If perpendiculars are drawn upon a diameter from each point of a given circle, then the locus of the points which divide these perpendiculars in a given ratio is an ellipse.

For every ordinate of the circle is altered in the same ratio.

10.21 Eccentric angle. Let P be a point on an ellipse and Q its corresponding point on the auxiliary circle. The angle ACQ which the line joining Q to the centre C of the ellipse makes with the major axis AC of the ellipse is called the *eccentric angle* of P and is generally denoted by 'ϕ'.

If PN be the ordinate of P,
$$CN=CQ\cos\phi=a\cos\phi,$$
and
$$PN=QN.\frac{b}{a}=CQ\sin\phi.\frac{b}{a}=b\sin\phi.$$

The coordinates of any point P on the ellipse are thus **($a \cos \phi$, $b \sin \phi$)** which is briefly called the point 'ϕ'.

10.22 Equation of the chord joining two points on an ellipse.

Let $(a \cos \phi, b \sin \phi)$, $(a \cos \phi', b \sin \phi')$ be two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation to the chord joining these points is

$$\begin{vmatrix} x & y & 1 \\ a \cos \phi & b \sin \phi & 1 \\ a \cos \phi' & b \sin \phi' & 1 \end{vmatrix}=0,$$

i.e., $\quad bx (\sin \phi-\sin \phi')+ay (\cos \phi'-\cos \phi)=ab \sin (\phi-\phi')$.

Dividing by $2ab \sin \frac{\phi-\phi'}{2}$, we get the equation to the required chord as

$$\frac{x}{a} \cos \frac{\phi+\phi'}{2}+\frac{y}{b} \sin \frac{\phi+\phi'}{2}=\cos \frac{\phi-\phi'}{2}.$$

Corollary. The equation of the tangent at $(a \cos \phi, b \sin \phi)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a} \cos \phi+\frac{y}{b} \sin \phi=1$.

Examples

1. *If the chord joining two points whose eccentric angles are α and β, cut the major axis of an ellipse at a distance c from the centre, show that*

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2}=\frac{c-a}{c+a},$$

where a is the semi-major axis of the ellipse.

Solution. Equation of the chord joining the points 'α' and 'β' on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is

$$\frac{x}{a} \cos \frac{\alpha+\beta}{2}+\frac{y}{b} \sin \frac{\alpha+\beta}{2}=\cos \frac{\alpha-\beta}{2}.$$

Let it cut the major axis in the point $(c, 0)$. Then

$$c=\frac{a \cos \frac{\alpha-\beta}{2}}{\cos \frac{\alpha+\beta}{2}}=a \frac{1+\tan \frac{\alpha}{2} \tan \frac{\beta}{2}}{1-\tan \frac{\alpha}{2} \tan \frac{\beta}{2}},$$

i.e., $$\frac{c-a}{c+a}=\tan \frac{\alpha}{2} \tan \frac{\beta}{2}.$$

2. If θ, ϕ be the eccentric angles of the extremities of a focal chord of an ellipse of eccentricity e, prove that

(*i*) $\pm e \cos \frac{1}{2} (\theta+\phi)=\cos \frac{1}{2} (\theta-\phi)$.

(*ii*) $\tan \theta \tan \frac{1}{2}\phi+\frac{1 \pm e}{1 \pm e}=0$

(*Lucknow*, 1964; *Agra*, 1967)

3. If any two chords be drawn through two points on the major axis of ellipse equidistant from the centre, show that

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \tan \frac{\delta}{2}=1,$$

where α, β, γ, δ are the eccentric angles of the extremities of the chords. (*Rajasthan*, 1967)

Hint. Use the result of Example 1.

4. *A point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose eccentric angle is α, is joined to the foci S and S', and PS, PS meet the curve again in Q and Q'. Show that the equation to QQ' is*

$$\frac{x}{a} \cos \alpha\ (1-e^2)+\frac{y}{b} \sin \alpha\ (1+e^2)=e^2-1,$$

e being the eccentricity of the ellipse.

(*Lucknow*, 1961; *I. A. S.*, 1963)

Solution. Let the eccentric angles of Q and Q' be ϕ and ϕ'. Then, from example 2 (*ii*),

$$\tan \frac{\phi}{2}=-\frac{1-e}{1+e} \cot \frac{\alpha}{2} \qquad \ldots(1)$$

and $$\tan \frac{\phi'}{2}=-\frac{1+e}{1-e} \cos \frac{\alpha}{2}. \qquad \ldots(2)$$

The equation to QQ' is

$$\frac{x}{a} \cos \frac{\phi+\phi'}{2}+\frac{y}{b} \sin \frac{\phi+\phi'}{2}=\cos \frac{\phi-\phi'}{2},$$

or $$\frac{x}{a}\left(1-\tan \frac{\phi}{2} \tan \frac{\phi'}{2}\right)+\frac{y}{b}\left(\tan \frac{\phi}{2}+\tan \frac{\phi'}{2}\right)=1+\tan \frac{\phi}{2} \tan \frac{\phi'}{2}.$$

Substituting for $\tan \frac{\phi}{2}$, $\tan \frac{\phi'}{2}$ from (1) and (2), we get the equation to QQ' in the desired form.

5. *Show that the sum of the eccentric angles of the points in which a circle cuts an ellipse is an even multiple of two right angles.*

Solution. Let $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the points in which the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is cut by a circle.

The equations of the chords joining the points 'α', 'β', 'γ' and 'δ' are

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2},$$

and
$$\frac{x}{a}\cos\frac{\gamma+\delta}{2}+\frac{y}{b}\sin\frac{\gamma+\delta}{2}=\cos\frac{\gamma-\delta}{2}.$$

The equations to a conic passing through the four points of intersection is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left(\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}-\cos\frac{\alpha-\beta}{2}\right)$$
$$\times\left(\frac{x}{a}\cos\frac{\gamma+\delta}{2}+\frac{y}{a}\sin\frac{\gamma+\delta}{2}-\cos\frac{\gamma-\delta}{2}\right)=0.$$

Since this is given to be a circle, the coefficient of xy in the above equation must be zero.

Hence,

$$\cos\frac{\alpha+\beta}{2}\sin\frac{\gamma+\delta}{2}+\cos\frac{\gamma+\delta}{2}\sin\frac{\alpha+\beta}{2}=0,$$

i.e., $\sin\frac{1}{2}(\alpha+\beta+\gamma+\delta)=0,$

from which, $\alpha+\beta+\gamma+\delta=2n\pi$

where n is an integer.

6. If three of the sides of a quadrilateral inscribed in an ellipse are parallel respectively to three given straight lines, show that the fourth side will also be parallel to a fixed straight line.

(*Allahabad*, 1960)

Hint. If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the vertices of the quadrilateral, $\frac{\alpha+\beta}{2}, \frac{\beta+\gamma}{2}, \frac{\alpha+\delta}{2}$ are constants, and $\frac{\gamma+\delta}{2}=\frac{\beta+\gamma}{2}+\frac{\alpha+\delta}{2}-\frac{\alpha+\beta}{2}$.

7. Show that the area of a triangle inscribed in an ellipse bears a constant ratio to the area of the triangle formed by joining points on the auxiliary circle corresponding to the vertices of the first triangle.

Hint. If $(a\cos\alpha, b\sin\alpha)$, $(a\cos\beta, b\sin\beta)$, $(a\cos\gamma, b\sin\gamma)$ be the vertices of the triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the points on the auxiliary circle corresponding to these are $(a\cos\alpha, a\sin\alpha)$ etc. The areas of the two triangles are

$$\frac{1}{2}\begin{vmatrix} a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \\ a\cos\gamma & b\sin\gamma & 1 \end{vmatrix} \text{ and } \frac{1}{2}\begin{vmatrix} a\cos\alpha & a\sin\alpha & 1 \\ a\cos\beta & a\sin\beta & 1 \\ a\cos\gamma & a\sin\gamma & 1 \end{vmatrix}.$$

10.3 Tangent and other loci. As in Chapter VI, or by regarding the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as a particular case of the general equation, we get the following loci.

The *tangent* at a point (x', y') on the ellipse is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

The *chord of contact* of tangents from (x', y') or the *polar* of (x', y') is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

The equation of the *pair of tangents* from (x', y') is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{a^2}-1\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right)^2.$$

The equation of the *chord whose middle point* is (x', y'), is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}.$$

10.31 Condition of tangency of the line y=mx+c.

Let the line $y=mx+c$ be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\phi, b\sin\phi)$.

The equation of the tangent at $(a\cos\phi, b\sin\phi)$ is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1.$$

This must be the same as the equation to the given straight line.

Therefore, comparing coefficients,

$$\frac{\sin\phi}{b}=\frac{\cos\phi}{-am}=\frac{1}{c},$$

from which, $c^2=a^2m^2+b^2.$

This then is the condition of tangency of the given line.

The line

$$\mathbf{y=mx+\sqrt{(a^2m^2+b^2)}}$$

is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whatever be the value of m.

The coordinates of the point of contact are

$$\left(\frac{-a^2m}{\sqrt{(a^2m^2+b^2)}},\ \frac{b^2}{\sqrt{(a^2m^2+b^2)}}\right).$$

Note 1. Corresponding to any given value of x and y, the equation $y=mx+\sqrt{(a^2m^2+b^2)}$ gives two values of m. This corroborates the fact that two tangents can be drawn to the ellipse from a given point.

Note 2. Since $c^2=a^2m^2+b^2$, we get two values of c corresponding to a given value of m. We can therefore draw two tangents to

$$y=mx+\sqrt{(a^2m^2+b^2)},\ y=mx-\sqrt{(a^2m^2+b^2)}$$

to the ellipse parallel to a given straight line.

These tangents are equidistant from the centre of the ellipse.

10.4 The Director circle. *The director circle is the locus of points the tangents from which to the ellipse are at right angles.*

Let (x', y') be a point on the director circle. The equation to the pair of tangents from (x', y') to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)=\left(\frac{xx'}{b^2}+\frac{yy'}{b^2}-1.\right)^2.$$

These are at right angle if the sum of the coefficients of x^2 and y^2 in the above equation is equal to zero.

Collecting these coefficients,

$$\frac{1}{a^2}\left(\frac{y'^2}{b^2}-1\right)+\frac{1}{b^2}\left(\frac{x'^2}{a^2}-1\right)=0,$$

i.e., $$x'^2+y'^2=a^2+b^2.$$

Hence the equation of the director circle is

$$\mathbf{x^2+y^2=a^2+b^2.}$$

Examples

1. A line $lx+my+n=0$ touches the ellipse $x^2/a^2+y^2/b^2=1$. Show that

$$a^2l^2+b^2m^2=n^2.$$

(*Agra*, 1967; *Magadh*, 1977)

2. Show that the line $x\cos\alpha+y\sin\alpha=p$ is a tangent to the ellipse $x^2/a^2+y^2/b^2=1$, if

$$p^2=a^2\cos^2\alpha+b^2\sin^2\alpha.$$ (*Gorakhpur*, 1971)

3. Prove that the tangents at the extremities of a diameter of an ellipse are parallel to one another.

4. The tangent at the point $(4\cos\theta, \frac{16}{\sqrt{11}}\sin\phi)$ to the ellipse $16x^2+11y^2=256$ is also tangent to the circle $x^2+y^2+2x=15$. Find the value of ϕ.
(*Roorkee*, 1966)
Ans. 120°.

5. Prove that the line joining two points on an ellipse, the difference of whose eccentric angles is constant, touches another ellipse.
(*Andhra*, 1963)

6. Tangents are drawn from any point on the ellipse $x^2/a^2+y^2/b^2=1$ to the circle $x^2 \cdot y^2=r^2$. Prove that the chords of contact are tangents to the ellipse $a^2x^2+b^2y^2=r^4$. (*Rajasthan*, 1966)

7. Find the locus of pole of tangents to the ellipse $x^2/a^2+y^2/b^2=1$ with respect to the parabola $y^2=4ax$.
Ans. $b^2y^2=4a^3(x^2-a^2)$.

8. Prove that the perpendicular from the focus of an ellipse whose centre is C on the polar of any point P will meet the line CP on the directrix.

9. In the ellipse $x^2/9+y^2/4=1$, find the equation of the chord which is bisected at the point $(2, -1)$. *Ans.* $8x-9y=25$.

10. If the difference between the eccentric angles of two points on the ellipse $x^2/a^2+y^2/b^2=1$, be $\frac{1}{2}\pi$, show that the tangents at these points intersect one another ellipse of the same eccentricity.
(*Roorkee*, 1962)

11. Chords of an ellipse are drawn through the positive end of the minor axis. Show that their middle points lie on another ellipse.

12. Obtain the locus of poles of tangents to the ellipse $x^2/a^2+y^2/b^2=1$ with respect to concentric ellipse

$$x^2/\alpha^2+y^2/\beta^2 \quad 1.$$ (*Magadh*, 1975)
Ans. $a^2x^2/\alpha^4+b^2y^2/\beta^4=1$.

13. The tangent at a point P of an ellipse meets the auxiliary circle in two points which subtend a right angle at the centre. Show that the eccentricity of the ellipse is $(1+\sin^2\phi)^{-1/2}$, where ϕ is the eccentric angle of the point P.

14. Find the equation of tangents at the ends of the latera recta of the ellipse $x^2/a^2+y^2/b^2=1$, and show that they pass through the intersection of the axis and the directrixes.

15. Find the locus of the middle points of portions of tangents to the ellipse $x^2/a^2+y^2/b^2=1$ included between the coordinate axes.

(*Agra*, 1962; *Magadh*, 1974)

Ans. $a^2/x^2+b^2/y^2=4$.

16. Chords of an ellipse pass through a fixed point. Prove that the locus of their middle points is an ellipse with its axes parallel to those of original axes.

17. If the product of perpendiculars from the foci of the ellipse $x^2/a^2+y^2/b^2=1$ upon the polar of P be always c^2, prove that the locus of P is the ellipse

$$b^4(c^2+a^2-b^2)x^2+c^2a^4y^2-a^4b^4.$$

18. If a triangle be inscribed in an ellipse and two of its sides are parallel to two given straight lines, prove that the envelope of the third side is another ellipse.

19. The perpendicular from the centre of an ellipse $x^2/a^2+y^2/b^2=1$ on the polar of a point with respect to the ellipse is constant and equal to c. Prove that the locus of the point is the ellipse

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}. \qquad (\textit{Kashmir}, 1974)$$

20. An ellipse slides between two straight lines at right angles to one another. Show that the locus of its centre is a circle.

Hint. If (h, k) be the coordinates of centre, $h^2+k^2=a^2+b^2$, since the origin lies on the director circle.

10.5 The Normal. The equation of the tangent at a point $(a\cos\phi, b\sin\phi)$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1.$$

The equation of the normal, which is perpendicular to the tangent at the point of contact, is

$$\frac{\sin\phi}{b}\left(x-a\cos\phi\right)-\frac{\cos\phi}{a}\left(y-b\sin\phi\right)=0,$$

i. e., $$\mathbf{ax\sec\phi-by\,cosec\,\phi=a^2-b^2.}$$

Now, let this normal pass through (h, k). Then,

$$ah\sec\phi-bk\,\text{cosec}\,\phi=a^2-b^2.$$

or
$$\frac{ah\left(1+\tan^2\frac{\phi}{2}\right)}{1-\tan^2\frac{\phi}{2}}-\frac{bk\left(1+\tan^2\frac{\phi}{2}\right)}{2\tan\frac{\phi}{2}}=a^2-b^2$$

or, arranging in powers of $\tan\frac{\phi}{2}$,

$$bk\tan^4\frac{\phi}{2}+2\left(ah+a^2-b^2\right)\tan^3\frac{\phi}{2}$$
$$+2\left(ah-a^2+b^2\right)\tan\frac{\phi}{2}-bk=0.$$

This is a fourth degree equation in $\tan\frac{\phi}{2}$, and gives four values of $\tan\frac{\phi}{2}$ for a given value of (h, k). If K be one of the four values.

$$\tan\frac{\phi}{2}=K.$$

i.e., $$\phi=2\tan^{-1}K.$$

and the general value of ϕ is equal to

$$2n\pi+2\tan^{-1}K.$$

This gives the same point on the ellipse as ϕ.

Thus corresponding to one root of the equation in $\tan\frac{\phi}{2}$, we get only one point on the ellipse.

Hence *for normal can be drawn from a point to an ellipse.*

Note. Unlike parabola, all the normals to the ellipse can be imaginary.

Examples

1. Prove that the line $lx+my+m+n=0$ is normal to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

if $$\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}. \qquad (\textit{Agra}, 1970)$$

2. *Show that locus of the middle point of normal chords of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is given by the equation*

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)=(a^2-b^2)^2.$$

(*Lucknow*, 1980, *Gorakhpur*, 1970, *Kashmir*, 1973)

Solution. Let (x', y') be the middle point of a normal chord of the given ellipse. The equation to the chord is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}. \qquad ...(1)$$

Also the equation of a normal to the ellipse is

$$ax \sec\phi - by \operatorname{cosec}\phi = a^2-b^2. \qquad ...(2)$$

Equations (1) and (2) should represent the same straight line. Hence, comparing coefficients,

$$\frac{x' \cos\phi}{a^3}=\frac{-y' \sin\phi}{b^3}=\frac{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}{a^2-b^2}$$

Solving for $\cos\phi$ and $\sin\phi$, squaring and adding, we get

$$\left(\frac{a^6}{x'^2}+\frac{b^6}{y'^2}\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)^2=(a^2-b^2)^2.$$

Hence the required locus is

$$\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=(a^2-b^2)^2.$$

3. Prove that the locus of the poles of normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

(*Lucknow*, 1978; *Agra*, 1970; *Magadh*, 1976)

4. The length of the major axis intercepted between the tangent and normal at a point 'ϕ' on the ellipse is equal to the semimajor axis. Prove that the eccentricity of the ellipse is

$$\{\sec\phi\,(\sec\phi-1)\}^{1/2}.$$

5. *Prove that the eccentricity of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is given by*

$$\frac{2\cot\omega}{\sin 2\theta}=\frac{e^2}{\sqrt{(1-e^2)}},$$

where ω *is one of the angles between the normals at the points whose eccentric angles are* θ *and* $\frac{\pi}{2}+\theta$.

(*Lucknow*, 1967)

Solution. The equations of the normals at the given points are

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2,$$

and

$$-ax\operatorname{cosec}\theta - by\sec\theta = a^2-b^2.$$

Therefore,
$$\tan\omega=\frac{\frac{a}{b}(\tan\theta+\cot\theta)}{\frac{a^2}{b^2}-1}$$

or
$$\frac{2\cot\omega}{\sin 2\theta}=\frac{a^2-b^2}{ab}=\frac{a^2e^2}{a^2\sqrt{(1-e^2)}}$$

$$=\frac{e^2}{\sqrt{(1-e^2)}}.$$

6. *If* $\alpha, \beta, \gamma, \delta$ *be the eccentric angles of four points on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *such that the normals at them are concurrent, then show that*

(*i*) $\alpha+\beta+\gamma+\delta$ is an old multiple of π.

(*ii*) $\Sigma\cos(\alpha+\beta)=0$.

(*iii*) $\Sigma\sin(\alpha+\beta)=0$. (*Gorakhpur*, 1969)

Solution. The equation of the normal at the point 'ϕ' is

$$ax\sec\phi - by\operatorname{cosec}\phi = a^2-b^2.$$

Let it pass through (h, k). Then,

$$ah\sec\phi - bk\operatorname{cosec}\phi = a^2-b^2. \qquad ...(1)$$

Put $e^{i\phi}=Z$. Then $e^{-i\phi}=\frac{1}{Z}$ and

$$2\cos\phi = Z+\frac{1}{Z},\quad 2i\sin\phi = Z-\frac{1}{Z}.$$

Substituting for ϕ in (1), and simplifying,

$$(a^2-b^2)Z^4-2Z^3(ah-ibk)+2Z(ah+ibk)-(a^2-b^2)=0 \qquad ...(2)$$

If Z_1, Z_2, Z_3, Z_4 be the roots of equation (2), then

$$\Sigma Z_1=\frac{2(ah-ibk)}{a^2-b^2} \qquad ...(3)$$

$$\Sigma Z_1Z_2=0 \qquad \ldots(4)$$

$$\Sigma Z_1Z_2Z_3=-\frac{2\,(ah+ibk)}{a^2-b^2} \qquad \ldots(5)$$

and $$Z_1Z_2Z_3Z_4=-1 \qquad \ldots(6)$$

If $\alpha, \beta, \gamma, \delta$ be the eccentric angles corresponding to Z_1, Z_2, Z_3, Z_4, then, from (6)

$$\sin(\alpha+\beta+\gamma+\delta)=0, \ \cos(\alpha+\beta+\gamma+\delta)=-1,$$

i.e., $\alpha+\beta+\gamma+\delta=(2n+1)\,\pi$, where n is an integer or zero.

From (4), we immediately get

$$\Sigma\cos(\alpha+\beta)=0, \ \Sigma\sin(\alpha+\beta)=0.$$

Note. The above results can also be obtained if we consider the equation in $\tan\frac{\phi}{2}$, *viz*, the equation

$$bk\tan\frac{\phi}{2}+2\,(ah+a^2-b^2)\tan^3\frac{\phi}{2}$$

$$+2\,(ah-a^2+b^2)\tan\frac{\phi}{2}-bk=0,$$

and take $\tan\frac{\alpha}{2}$, $\tan\frac{\beta}{2}$, $\tan\frac{\gamma}{2}$, $\tan\frac{\delta}{2}$ as the four roots of this equation. The student should try this as an exercise.

7. *If the normals at the points whose eccentric angles are α, β, γ, are concurrent, then*

$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0. \qquad (\textit{Lucknow}, 1978)$$

Solution. If δ be the eccentric angle of the fourth point, the normal at which meets the normals at 'α', 'β', 'γ' at their common point of intersection, we have from the preceding example,

$$\alpha+\beta+\gamma+\delta=(2n+1)\,\pi \qquad \ldots(1)$$

and $$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)$$
$$+\sin(\alpha+\delta)+\sin(\beta+\delta)+\sin(\gamma+\delta)=0. \qquad \ldots(2)$$

Eliminating δ from (2) with the help of (1)

$$2\,\{\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)\}=0,$$

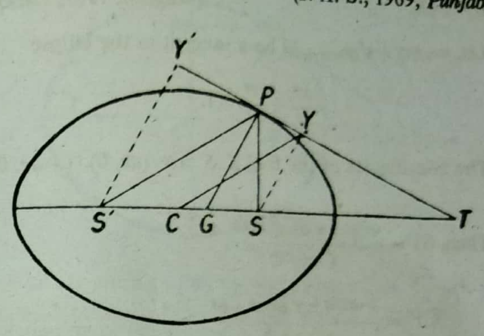
or $$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0.$$

8. Prove that if $\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0$, the normals at the points whose eccentric angles are α, β, γ will be concurrent.

10.6 Propositions on the ellipse. We shall prove below some propositions on the ellipse.

(*i*) *The tangent and normal at any point of an ellipse bisect the external and internal angles between the focal radii to the point.*

(*I. A. S.*, 1969; ***Punjab*, 1975**)



Let P be a point (x', y') on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let the tangent and normal at P meet the major axis in T and G respectively. The equation of the tangent at P is $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$. The equation to the normal at P is therefore

$$\frac{x'}{a^2}\,(y-y')-\frac{y'}{b^2}\,(x-x')=0.$$

Putting $y=0$ in this equation,

$$x=\frac{x'}{a^2}\,(a^2-b^2)=e^2x',$$

which is the value of CG.

Now, $SG=CS-CG$
$=ae-e^2x'$
$=e\,(a-ex')$
$=e.PS.$ (§ 10.15)

Similarly, $GS'=e.\,PS'$.

Hence $\frac{PS}{PS'}=\frac{GS}{GS'}$,

i.e., PG bisects the angle SPS'.

The tangent PT being perpendicular to the normal PG bisects the external angle between PS and PS', the focal radii of P.

(ii) *If SY, and $S'Y'$ be perpendiculars from the foci upon the tangent at any point of the ellipse, then $SY.S'Y'=b^2$ and Y, Y' lie on the auxiliary circle.* (*Magadh*, 1977; *Lucknow*, 1966)

Let $y=mx+\sqrt{a^2m^2+b^2}$ be a tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The coordinates of the foci S, S' are $(ae, 0)$, $(-ae, 0)$ respectively.

Then, $SY=\dfrac{aem+\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}$

and $S'Y'=\dfrac{-aem+\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}$

Hence $S'Y.S'Y'=\dfrac{a^2m^2+b^2-a^2e^2m^2}{1+m^2}$,

$$=\frac{b^2(1+m^2)}{1+m^2},$$

$$=b^2.$$

The equation to SY is

$$my+x=ae \qquad ...(1)$$

Now Y lies on (1), and the tangent

$$y-mx=\sqrt{a^2m^2+b^2}$$

Squaring and adding (1) and (2), ...(2)

$$(x^2+y^2)(1+m^2)=a^2m^2+b^2+a^2e^2$$

$$=a^2(1+m^2)$$

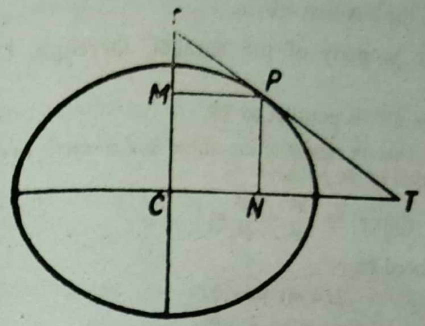
i.e., $x^2+y^2=a^2$.

Y therefore lies on the auxiliary circle. Similiarly Y' lies on the auxiliary circle.

(iii) *If the tangent at any point P meets the major axis in T and the minor axis in t, then*

$$CN.CT=a^2,\ CM.Ct=b^2,$$

N and M being the feet of the perpendiculars from P on the respective axes.



Let P be the point (x', y') on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The equation of the tangent at P is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

Putting $y=0$ in this equation,

$$xx'=a^2.$$

But $x=CT$ and $x'=CN$.

Hence $CT.CN=a^2$.

Similarly, $Ct.CM=b^2$.

(iv) *If SY and $S'Y'$ are perpendiculars from the foci upon a tangent to the ellipse, then CY and CY' are parallel to $S'P$ and SP respectively.*

From the preceding proposition

$$CT=\frac{a^2}{CN}.$$

Also, $S'T=CS'+CT=ae+\dfrac{a^2}{CN}$

$$=\frac{a(a+e.CN)}{CN}$$

$$=\frac{a.S'P}{CN}.$$

Therefore, $\frac{S'T}{CT}=\frac{S'P}{a}=\frac{S'P}{CY}$,

since Y lies on the auxiliary circle.

From the property of the parallel, therefore, CY is parallel to $S'P$.

Similarly, CY' is parallel to SP.

(v) *The common chords of an ellipse and a circle are equally inclined to the axes of the ellipse.*

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

and let the chord be

$$lx+my+n=0 \quad \text{...(1)}$$

and

$$l'x+m'y+n'=0. \quad \text{...(2)}$$

The equation to the conic through the four points in which (1) and (2) meet the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\,(lx+my+n)\,(l'x+m'y+n')=0.$$

Since this is a circle, the coefficient of xy in this equation must vanish.

We thus have

$$lm'+ml'=0,$$

i.e.,

$$\frac{l}{m}=-\frac{l'}{m'},$$

This proves the proposition.

Examples

1. Prove that the subtangent and subnormal of a point (x_1, y_1) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are respectively $\left(\frac{a^2}{x_1}-x_1\right)$ and $\frac{b^2x_1}{a}$.

2. Show that the eccentricity of the ellipse in which the normal at one end of a latus rectum passes through an end of the minor axis is given by the equation

$$e^4+e^2=1. \qquad \textit{(Kashmir, 1975)}$$

3. Q is the point on the auxiliary circle corresponding to P on the ellipse. PLM is drawn parallel to the radius CQ to meet the axes in L and M. Prove that PM and PL are equal to the semi-axes.

4. Prove that in an ellipse the sum of the squares of the perpendiculars on any tangent from two points on the minor axis, each distant $\sqrt{a^2-b^2}$ from the centre, is $2a$. *(Kashmir, 1971)*

5. If p be the length of the perpendicular from a focus upon the tangent at any point P of the ellipse and r the distance of P from the focus, prove that

$$\frac{b^2}{p^2}=\frac{2a}{r}-1.$$

6. The ordinate NP of a point P on an ellipse is produced to meet the tangent at an end of the latus rectum through the focus S in R. Prove that $NR=SP$.

7. The normal at any point P of an ellipse cuts the major axis in G. Show that as P varies the middle point of PG traces another ellipse.

8. Show that the locus of the foot of the perpendicular from the centre on any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve

$$(x^2+y^2)^2=a^2x^2+b^2y^2 \qquad \textit{(Rajastan, 1973)}$$

9. A circle is described on a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ lying on the straight line $p\frac{x}{a}+q\frac{y}{b}=1$ as diameter. Prove that the equation of the straight line joining the other two common points of the ellipse and the circle is

$$p\frac{x}{a}-q\frac{y}{b}=\frac{a^2+b^2}{a^2-b^2}.$$

Hint. From proposition (v) (§ 10.6), the equation to the other common chord is

$$\frac{px}{a}-\frac{qy}{b}=k.$$

The equation to circle is then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left(\frac{px}{a}+\frac{qy}{b}-1\right)\left(\frac{px}{a}-\frac{qy}{b}-k\right)=0,$$

where

$$\frac{1}{a^2}(1+\lambda b^2)=\frac{1}{b^2}(1-\lambda q^2),$$

i.e.,

$$\lambda=\frac{a^2-b^2}{b^2p^2+a^2q^2}.$$

The centre of the circle lies on the given chord, which determines k.

10. Tangent drawn from a point P to a given ellipse meet a given tangent whose point of contact is O in Q, Q'. Prove that if the distance of P from the given tangent be constant, the rectangle $OQ \cdot OQ'$ will be constant.

11. *If ABC be a maximum triangle inscribed in an ellipse show that the eccentric angles of the vertices differ by* $\frac{2\pi}{3}$ *and that the normals at A, B, C are concurrent.*

Solution. Let α, β, γ be the eccentric angles of the points A, B, C lying on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If A'. B'. C' be the coorresponding points on the auxiliary circle, then from example 7 of § 10. 22

$$\Delta ABC=\frac{b}{a}\Delta A'B'C'.$$

The two areas are therefore greatest simultaneously.

$\Delta A'B'C'$ being in a circle is greatest when it is equilateral. In that case

$$\alpha\sim\beta=\beta\sim\gamma=\gamma\sim\alpha=\frac{2\pi}{3}.$$

Hence the triangle ABC is maximum when the eccentric angles of the vertices differ by $\frac{2\pi}{3}$.

Now,
$$\alpha+\beta=2\alpha+\frac{2\pi}{3},$$
$$\beta+\gamma=2\alpha+\frac{6\pi}{3},$$
$$\gamma+\alpha=2\alpha+\frac{4\pi}{3}.$$

Hence

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)$$
$$=\sin\left(2\alpha+\frac{2\pi}{3}\right)+\sin 2\alpha+\sin\left(2\alpha+\frac{4\pi}{3}\right)$$
$$=\sin 2\alpha-\sin 2\alpha$$
$$=0,$$

which proves the second part.

10. 7 Conjugate diameters. In Chapter VI we defined conjugate diameters and obtained the condition that two diameters of the conic represented by the general equation of the second degree be conjugate. We shall now derive this condition independently for the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

To recapitulate, *two diameters of the ellipse will be conjugate when each bisects chord parallel to the other.*

Let $y=mx$ and $y=m'x$ be two conjugate diameters of the ellipse.

Let (x', y') be the middle point of a chord of the ellipse drawn parallel to $y=mx$.

The equation to this chord is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}.$$

Hence,

$$m=-\frac{b^2}{a^2}\frac{x'}{y'}.$$

The locus of (x', y') is therefore the diameter

$$y=-\frac{b^2}{a^2m}x.$$

Since this is the same as $y=m'x$,

$$m'=-\frac{b^2}{a^2m}$$

or,
$$mm'=-\frac{b^2}{a^2}.$$

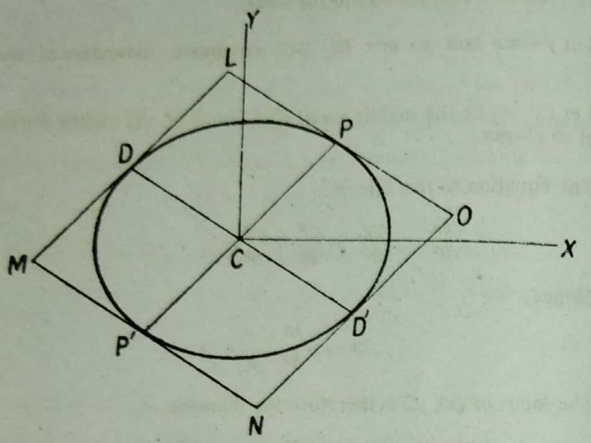
The above relation does not change if m and m' are interchanged. *The two diameters $y=mx$ and $y=m'x$ are therefore conjugate if*

$$mm'=-\frac{b^2}{a^2}.$$

10.71 Eccentric angles of the extremities of two conjugate diameters.

Let PCP', DCD' be two conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let θ, ϕ be the eccentric angles of P and D. The slopes of CP and CD are then

$$\frac{b\sin\theta}{a\cos\theta} \text{ and } \frac{b\sin\phi}{a\cos\phi}.$$



Hence, from the preceding article,

$$\frac{b \sin \theta}{a \cos \theta} \frac{b \sin \phi}{a \cos \phi} = -\frac{b^2}{a^2}$$

or $$\cos \theta \cos \phi + \sin \theta \sin \phi = 0,$$

i.e., $$\cos (\theta - \phi) = 0.$$

Hence, $\theta \sim \phi$ is an odd multiple of $\frac{\pi}{2}$.

If, therefore, the coordinates of an extremity of a diameter of the ellipse be ($a \cos \phi$, $b \sin \phi$), the coordinates of an extremity of the conjugate diameter are ($-a \cos \phi$, $b \cos \phi$).

10.72 Sum of the squares of conjugate semi-diameters.

Let CP, CD be two conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let the coordinates of P be ($a \cos \phi$, $b \sin \phi$). The coordinates of D are then ($-a \sin \phi$, $b \cos \phi$).

Therefore, $CP^2 = a^2 \cos^2 \phi + b^2 \sin^2 \phi$,

$CD^2 = a^2 \sin^2 \phi + b^2 \cos^2 \phi$

Adding, $CP^2 + CD^2 = a^2 + b^2$.

The sum of the squares of conjugate semi-diameters of an ellipse is therefore constant and equal to the sum of the squares of the semi-axes of the ellipse, which are a particular case of conjugate semidiameters.

10.73 Area of the parallelogram formed by tangents at the extremities of two conjugate diameters.

Let PCP', DCD' be two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The coordinates of P and D can be written as ($a \cos \phi$, $b \sin \phi$) and ($-a \sin \phi$, $b \cos \phi$). The coordinates of P' and D', the other extremities of these diameters, are ($-a \cos \phi$, $-b \sin \phi$) and ($a \sin \phi$, $-b \cos \phi$).

The equation of the tangent at P is

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1,$$

and the equation of CD is

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 0.$$

The tangent at P is therefore parallel to CD.

The equation of the tangent at P' being

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = -1$$

we see that this is also parallel to CD.

Similarly, the tangents at D, D' will be seen to be parallel to PC.

The tangents at the four points P, P', D, D' thus form a parallelogram $LMNO$, of which the area is four times the area of the parallelogram $LPCD$.

Now, the area of the parallelogram $LPCD$.

$= CD \times$ Perpendicular from C upon the tangent at P

$$= \sqrt{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)} \times \frac{1}{\sqrt{\left(\frac{\cos^2 \phi}{a^2} + \frac{\sin^2 \phi}{b^2}\right)}}$$

$= ab.$

The area of the parallelogram $LMNO$ is therefore equal to $4ab$.

We thus conclude that the area of the parallelogram formed by the tangents at the extremities of two conjugate diameters of an ellipse is constant and equal to the product of the axes.

10.74 Equi-conjugate diameters. *Two conjugate diameters are said to be equi-conjugate when they are equal in length.*

The diameters PCP', DCD' are equi-conjugate if

$$a^2 \cos^2 \phi + b^2 \sin^2 \phi = a^2 \sin^2 \phi + b^2 \cos^2 \phi,$$

i.e., if $$(a^2 - b^2) \cos 2\phi = 0.$$

Since a and b are unequal, this gives

$$\cos 2\phi = 0,$$

i.e., $$\phi = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

Taking $\phi = \frac{\pi}{4}$, the equation to CP is

$$y = \left(\frac{b}{a} \tan \phi \right) x$$

$$= \frac{b}{a} x.$$

The equation to CD is $y = -\frac{b}{a} x$.

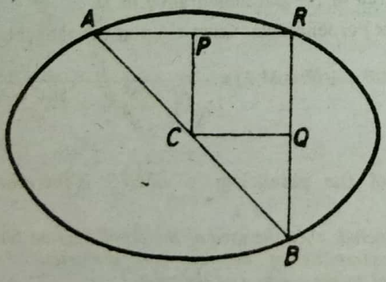
Corollary. The equi-conjugate diameters of an ellise lie along the diagonals of the rectangle formed by the tangent at the ends of the major and minor axes.

10.75 Conjugate diameters and conjugate lines. *We shall show that the conjugate diameters are a special case of conjugate lines.*

We know that the polar of a point and the chord of contact of tangents drawn from it to the ellipse are the same thing. The pole of the diameter PCP' will thus be the point of intersection of the tangents at P and P'. These tangents beings parallel, the pole lies at infinity on the conjugate diameter DCD'.

Similarly the pole of the diameter DCD' lies on PCP'. PCP' and DCD' are therefore conjugate lines (See Chapter VI for the definition of conjugate lines).

10.76 Supplemental chords.



Definition. Chords joining any point on an ellipse to the extremities of a diameter are called supplemental chords.

We shall prove the following theorem :

The supplemental chords of an ellipse are parallel to conjugate diameters.

Let C be the cetre of an ellipse and R a point on it. Let ABC be a diameter of the ellipse.

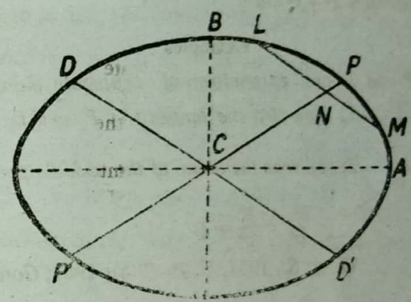
Join AR, BR and let P, Q be their middle points.

Now, PC is parallel to BR. The diameter parallel to BR therefore bisects AR and consequently all chords are parallel to AR. Similarly, the diameter lying along CQ bisects all chords parallel to BR. CP and CQ are thus lines along which two conjugate diameters of the ellipse lie.

This proves the theorem.

10.77 Ellipse referred to two conjngate diameters.

We shall now find the equation to the ellipse when two conjugate diameters are taken as coordinate axes.



Let (x, y) be the coordinates of a point on the ellipse when its axes CA and CB are taken as coordinate axes, and let (X, Y) be the coordinates of the same point when the conjugate diameters PCP', DCD' are taken as coordinate axes.

From Chapter IV we see that the tranformation relation will be of the form

$$x = \lambda X + \mu Y, \quad y = \lambda' X + \mu' Y,$$

where λ, μ, λ', μ' depend on the angles ACP, PCD.

Referred to cojugate diameters, the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

therefore becomes $$AX^2 + 2HXY + BY^2 = 1. \quad ...(1)$$

Let CP and CD be of length a' and b' respectively. The coordinates of P and D are thus $(a', 0)$ and $(0, b')$.

Since these points lie on the ellipse,

$$Aa'^2=1,\ Bb'^2=1,$$

i.e.,

$$A=\frac{1}{a'^2},\ B=\frac{1}{b'^2}.$$

Let LM be a chord of the ellipse, drawn parallel to DCD', and let it meet CP in N. Then N is the middle point of LM.

If therefore L is the point (X, Y), M is the point $(X, -Y)$.

Since both L and M lie on the ellipse, we have, substituting their coordinates in equation (1) and subtracting

$$H=0.$$

The equation of the ellipse referred to conjugate diameters of lengths $2a'$ and $2b'$ is therefore

$$\frac{X^2}{a'^2}+\frac{Y^2}{b'^2}=1.$$

Corollary. The equation of an ellipse referred to two equi-conjugate diameters $x^2+y^2=a'^2$.

Examples

1. *If P and D are extremities of conjngate diameters of an ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *show that the tangents at P and D meet on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$, *and that the locus of the midddle point of PD* is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\tfrac{1}{2}.$$

(*I. A. S.*, 1977; *U. P. C. S.*, 1967; *Gorakhpur*, 1974)

Solution.

Part I : Let P be the point $(a\cos\phi,\ b\sin\phi)$. Then D is the point $(-a\sin\phi, b\cos\phi)$.

The equations of tangents at P and D are respectively

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1,$$

and

$$-\frac{x}{a}\sin\phi+\frac{y}{b}\cos\phi=1.$$

Squaring and adding, the locus of the intersection of these tangents is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2.$$

Part II : Let (x, y) be the middle point of PD. Then

$$2x=a(\cos\phi-\sin\phi),$$

and

$$2y=b(\sin\phi+\cos\phi).$$

From these, $4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=2,$

or

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\tfrac{1}{2}.$$

2. P and D are the extremities of a pair of conjugate radii of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Show that PD touches the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\tfrac{1}{2}.$$

Hint. The equation of PD is

$$\frac{x}{a}\cos\left(\phi+\frac{\pi}{4}\right)+\frac{y}{b}\sin\left(\phi+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}.$$

3. If there is a system of parallel chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that the sum of the eccentric angles of the ends of each chord is same. If the constant value of the sum of the eccentric angles is λ, find the equation of the diameter of the ellipse, which bisects the system of parallel chords. (*Allahabad*, 1962)

Ans. $y=\frac{b}{a}x\tan\frac{\lambda}{2}$.

4. *CP, CQ' are conjugate semi-diameters of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the circles with *CP* and *CQ as diameters intersect at R, show that R lies on the curve*

$$2(x^2+y^2)^2=a^2x^2+b^2y^2. \quad (\textit{Gorakhpur}, 1975)$$

Solution. Let P and Q be the points $(a\cos\phi,\ b\sin\phi)$, $(-a\sin\phi, b\cos\phi)$.

The equations of the circles with CP and CQ as diameters are

$$x(x-a\cos\phi)+y(y-b\sin\phi)=0,$$

and

$$x(x+a\sin\phi)+y(y-b\cos\phi)=0.$$

Writing these as

$$x^2+y^2=ax\cos\phi+by\sin\phi,$$

and

$$x^2+y^2=-ax\sin\phi+by\cos\phi.$$

Squaring and adding, the locus of R is

$$2(x^2+y^2)^2=a^2x^2+b^2y^2.$$

5. CP, CQ are conjugate diameters of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. A tangent parallel to PQ meets CP, CQ at R, S; show that R and S lie on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2. \qquad \textit{(Gorakhpur, 1973)}$$

6. In an ellipse a pair of conjugate diameters is produced to meet the directrix. Show that the orthocentre of the triangle so formed is the focus. (*Nagpur*, 1974)

7. *If PCP', DCD' be two conjugate diameters of an ellipse and if QVQ' be a double ordinate of the diameter CP, prove that*

$$\frac{QV^2}{CD^2}=\frac{PV.VP'}{CP^2}.$$

Solution. Taking PCP', DCD' as coordinate axes,

$$\frac{CV^2}{CP^2}+\frac{QV^2}{CD^2}=1.$$

Therefore,

$$\frac{QV^2}{CD^2}=1-\frac{CV^2}{CP^2}=\frac{(CP+CV)(CP-CV)}{CP^2}$$
$$=\frac{PV.VP'}{CP^2}.$$

8. QVQ' is a double ordinate of the diameter CP, and the tangent Q meets CP in T. Prove that $CV.CT=CP^2$.

Solution. If Q be the point (x_1, y_1), the tangent at Q is $\frac{Xx_1}{a'^2}+\frac{Yy_1}{b'^2}=1$. Putting $Y=0$, $X=\frac{a'^2}{x_1}$, *i.e.*, $CT=\frac{CP^2}{CV}$.

9. *PCP' and DCD' are conjugate diameters of an ellipse, and ϕ is the eccentric angle of P. Prove that $\frac{1}{2}\pi-3\phi$ is the eccentric angle of the point where the circle $PP'D$ again cuts the ellipse.*

Solution. The eccentric angles of D and P' are $\frac{\pi}{2}+\phi$, $\pi+\phi$.

If α be the eccentric angle of the point where the circle $PP'D$ again cuts the ellipse,

$$\phi+\left(\frac{\pi}{2}+\phi\right)+(\pi+\phi)+\alpha=2n\pi$$

i.e.,
$$\alpha=2n\pi-\frac{3\pi}{2}-3\phi.$$

Putting $n=1$, $\qquad \alpha=\frac{\pi}{2}-3\phi.$

Any other value of n gives the same point on the ellipse.

10. If PCP' and DCD' are a pair of conjugate diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and Q is any point on the circle

$$x^2+y^2=c^2,$$

show that

$$PQ^2+DQ^2+P'Q^2+D'Q^2=2(a^2+b^2+2c^2). \qquad \textit{(Kashmir, 1956)}$$

10.8 Miscellaneous Solved Examples.

1. *Show that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angles differ by a constant is another ellipse.*

Solution. The equations of tangents at 'θ', 'ϕ' are

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1,$$

and
$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1.$$

If (h, k) be the point of intersection,

$$h=a\frac{\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}, k=b\frac{\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}.$$

Eliminating $\theta+\phi$, the required locus is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\sec^2\tfrac{1}{2}(\theta-\phi)=c^2$$

where c^2 is constant.

2. *Find the locus of the intersection of normals at the extremities of a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ drawn parallel to a given straight line.*

Solution. Let 'θ', 'ϕ' be the extermities of the chord. The equations of the normals at these points are

$$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2,$$
$$ax\sec\phi-by\operatorname{cosec}\phi=a^2-b^2,$$

If (h, k) be the point of intersection of these normals,

$$h=\frac{a^2-b^2}{a}\cos\theta\cos\phi\,\frac{\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}} \qquad \ldots(1)$$

and
$$k=-\frac{a^2-b^2}{a}\sin\theta\sin\phi\,\frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}} \qquad \ldots(2)$$

Since the slope of the chord is $-\frac{b}{a}\cot\frac{\theta+\phi}{2}$, by the condition of the question $\theta+\phi=2\alpha$, wher α is a constant.

From (1) and (2),

$$\frac{ah}{\cos\alpha}+\frac{bk}{\sin\alpha}=(a^2-b^2)\frac{\cos 2\alpha}{\cos\frac{\theta-\phi}{2}},$$

and
$$\frac{ah}{\cos\alpha}-\frac{bk}{\sin\alpha}=(a^2-b^2)\frac{\cos(\theta-\phi)}{\cos\frac{\theta-\phi}{2}}$$

$$=(a^2-b^2)\left(2\cos\frac{\theta-\phi}{2}-\sec\frac{\theta-\phi}{2}\right).$$

Eliminating $\theta-\phi$ from these, the locus of (h, k) is

$$a^2x^2+2abxy\operatorname{cosec}2\alpha+b^2y^2=(a^2-b^2)^2\cos^2 2\alpha.$$

3. *Show that the feet of the normals from a point to an ellipse lie on a conic which passes through the given point and the centre of the ellipse.* (*Gorakhpur*, 1973)

Solution. Let the equation to the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let (x', y') be a point on it. The equation to the normal at (x', y') is

$$\frac{x'}{a^2}(y-y')-\frac{y'}{b^2}(x-x')=0.$$

If (h, k) be the given point,

$$\frac{x'}{a^2}(k-y')-\frac{y'}{b^2}(h-x')=0,$$

since the normal passes through (h, k).

The locus of (x', y') is therefore the conic

$$xy\left(\frac{1}{b^2}-\frac{1}{a^2}\right)-\frac{hy}{b^2}+\frac{kx}{a^2}=0.$$

The conic obviously passes through $(0, 0)$ which is the centre of the ellipse and the given point (h, k).

The conic will be seen to be a rectangular hyperbola which we shall study in the next Chapter.

4. *If ω is the difference of the eccentric angles of two points on an ellipse the tangents at which are at right angles, prove that $ab\sin\omega=\lambda\mu$, where λ, μ are the semi-diameters parallel to the tangent at the points and a, b are semi-axes of the ellipse.* (*U. P. C. S.*, 1954)

Solution. Let P, Q be two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and let θ, ϕ be their eccentric angles.

The diameter parallel to the tangent at P will be conjugate to the diameter CP, and therefore one of its extremities will be $(-a\sin\theta, b\cos\theta)$.

$$\lambda^2=a^2\sin^2\theta+b^2\cos^2\theta.$$

Similarly, $\mu^2=a^2\sin^2\phi+b^2\cos^2\phi$.

$$\lambda^2\mu^2=(a^2\sin^2\theta+b^2\cos^2\theta)(a^2\sin^2\phi+b^2\cos^2\phi)$$
$$=(a^2\sin\theta\sin\phi+b^2\cos\theta\cos\phi)^2+a^2b^2(\sin\theta\cos\phi-\cos\theta\sin\phi)^2$$
$$=a^2b^2\sin^2(\theta-\phi),$$

since $a^2\sin\theta\sin\phi+b^2\cos\theta\cos\phi=0$ is the condition that the tangents at P and Q be perpendicular.

Hence, $\lambda\mu=ab\sin(\theta\sim\phi)$

$=ab\sin\omega$

5. *A parallelogram circumscribes the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, and two of its angular points are on the lines $x^2-h^2=0$; prove that the other two are on the conic*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\left(1-\frac{a^2}{h^2}\right)-1=0. \qquad (\textit{Agra}, 1960)$$

Solution. The equations to the pairs of tangents from (h, k_1) and $(-h, k_2)$ to the ellipse are

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k_1^2}{b^2}-1\right)=\left(\frac{xh}{a^2}+\frac{yk_1}{b^2}-1\right)^2$$

and
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k_2^2}{b^2}-1\right)=\left(-\frac{xh}{a^2}+\frac{yk_2}{b^2}-1\right)^2$$

Since the two pairs are parallel, the coefficients of x^2, y^2 and xy in these equations must be proportional.

This gives $$k_2=-k_1=k, \text{ say}.$$

The equations to the sides of the parallelogram are thus

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)=\left(\frac{xh}{a^2}+\frac{yk}{b^2}-1\right)^2 \quad \ldots(1)$$

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)=\left(\frac{xh}{a^2}+\frac{yk}{b^2}+1\right)^2 \quad \ldots(2)$$

Taking the difference of (1) and (2),

$$\frac{xh}{a}+\frac{yk}{b^2}=0. \quad \ldots(3)$$

Hence, from (1) or (2),

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right. \quad \ldots(4)$$

Eliminating k between (3) and (4),

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{x^2h^2b^2}{a^4y^2}-1\right)=1,$$

or $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{a^2y^2}{b^2h^2}\right)=\frac{a^2y^2}{b^2h^2},$$

or $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{x^2}{a^6}+\frac{y^2}{b^2}-\frac{a^2y^2}{b^2h^2}\right)=0,$$

i.e., $$\frac{x^2}{a^2}+\frac{y^2}{b^2}\left(1-\frac{a^2}{h^2}\right)-1=0,$$

which proves the proposition.

6. *Prove that the coordinates of the centre of the circle through points 'α', 'β', 'γ' of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *are*

$$\left\{\frac{a^2-b^2}{4a}(\Sigma\cos\alpha+\cos\Sigma\alpha),\ -\frac{a^2-b^2}{4b}(\Sigma\sin\alpha-\sin\Sigma\alpha)\right\}.$$

Solution. Let the equation to the circle be

$$x^2+y^2+2gx+2fy+c=0,$$

and let 'θ' be any of the given points.

The eccentric angles of the points in which the ellipse is cut by the circle are the roots of the equation

$$a^2\cos^2\theta+b^2\sin^2\theta+2ag\cos\theta+2bf\sin\theta+c=0,$$

i.e., $$\{\cos^2\theta\,(a^2-b^2)+2ag\cos\theta+b^2+c\}^2=4b^2f\,(1-\cos^2\theta).$$

If δ be the eccentric angle of the fourth point of intersection,

$$\cos\alpha+\cos\beta+\cos\gamma+\cos\delta=-\frac{4ag}{a^2-b^2}. \quad \ldots(1)$$

Similarly, $$\sin\alpha+\sin\beta+\sin\gamma+\sin\delta=\frac{-4bf}{b^2-a^2} \quad \ldots(2)$$

Also, from Example 5 of § 10.22,

$$\alpha+\beta+\gamma+\delta=2n\pi. \quad \ldots(3)$$

From (1) and (3),

$$-g=\frac{a^2-b^2}{4a}\{\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)\}$$

From (2) and (3),

$$-f=-\frac{a^2-b^2}{4b}\{\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)\}$$

which proves the proposition.

7. *If the normals of four points of the ellipse $x^2/a^2+y^2/b^2=1$ are concurrent, and if two of the points lie on the line $lx+my=1$, show that the other two points lie on the line*

$$\frac{x}{a^2l}+\frac{y}{b^2m}+1=0.$$

Hence, show that if the feet of the two of the normals from a point P to the ellipse $x^2/a^2+y^2/b^2=1$ are coincident, the locus of the middle point of the chords joining the feet of the other normals is

$$\left(\frac{xy}{ab}\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^3.$$

Solution. Let the feet of the two remaining normals lie on

$$\alpha x+\beta y+1=0. \quad \ldots(1)$$

Then the feet of the four normals lie on

$$(lx+my-1)(\alpha x+\beta y+1)=0.$$

or $$x^2\alpha l+y^2\beta m+xy(\alpha m+\beta l)-(\alpha x+\beta y+1)=0. \quad \ldots(2)$$

These feet of normals also lie on the given ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \quad \ldots(3)$$

Comparing coefficients of x^2 and y^2, we obtain

$$\alpha=\frac{1}{a^2l}\text{ and }\beta=\frac{1}{b^2m}.$$

Putting the values of α and β in (1), the equation of the line joining the feet of the other two normals is

$$\frac{x}{a^2 l}+\frac{y}{b^2 m}+1=0. \qquad \ldots(4)$$

Further, if the first two feet of the normals coincide, the line $lx+my=1$ must become tangent to the ellipse. Consequently,

$$a^2l^2+b^2m^2=1. \qquad \ldots(5)$$

Let (x_1, y_1) be the middle point of (4). Then this equation is the same as

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}. \qquad \ldots(6)$$

Since (4) and (6) represent the same line, we obtain, on comparing coefficients of x, y and constant terms,

$$l=\frac{-1}{x_1}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right), m=\frac{-1}{y_1}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right).$$

Putting these values for l and m in (5),

$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)^2\left(\frac{a^2}{x_1^2}+\frac{b^2}{y_1^2}\right)=1$$

or

$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)^3=\left(\frac{x_1^2/a^2+y_1^2/b^2}{\frac{a^2}{x_1^2}+\frac{b^2}{y_1^2}}\right)$$

$$=\left(\frac{x_1y_1}{ab}\right)^2$$

Consequently, the locus of (x_1, y_1) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^3=\left(\frac{xy}{ab}\right)^2.$$

Examples on Chapter X

1. Prove that the least value of the intercept of a tangent to an ellipse between the coordinate axes is equal to the sum of the semi-axis.

2. Find the locus of the middle points of the chords of the ellipse $x^2/a^2+y^2/b^2=1$, which subtend a right angle at the centre.

$$Ans.\ \frac{x^2}{a^4}+\frac{y^2}{b^4}=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$$

3. From a point on the circle $x^2+y^2=a^2$, tangents are drawn to the ellipse $x^2/a^2+y^2/b^2=1$. Show that the locus of the middle points of the chords of contract is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2}. \qquad (Lucknow,\ 1966)$$

4. Show that the tangents of the extremities of all chords of the ellipse $x^2/a^2+y^2/b^2=1$, which subtend a right angle at the centre intersect on the ellipse

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}.$$ (*Lucknow*, 1968; *I.A.S.*, 1970)

5. Prove that the line joining the extremities of any two diameters of an ellipse which are of right angles to one another, will always touch a fixed circle.

6. Show that the line $lx+my+n=0$ will cut the ellipse $x^2/a^2+y^2/b^2=1$ in points whose eccentric angles differ by $\frac{1}{2}\pi$, if

$$a^2l^2+b^2m^2=2n^2.$$ (*Jiwaji*, 1967)

7. The normals of four points on the ellipse $x^2/a^2+y^2/b^2=1$ meet in the points (h, k). Prove that the mean position of the four points is

$$\left[\frac{1}{4}\cdot\frac{a^2b}{a^2-b^2}, \frac{1}{4}\cdot\frac{ab^2}{a^2-b^2}\right]$$ (*Gorakhpur*, 1972)

8. To any point P of an ellipse $x^2/a^2+y^2b^2=1$, corresponds a point Q on a auxiliary circle. The normal to the ellipse of P and the circle at Q meet in R. Prove that the locus of R is

$$x^2=y^2=(a+b)^2.$$

9. If P and D be the ends of conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$, show that the locus of foot of perpendicular from the centre of the ellipse on PD is

$$2(x^2+y^2)^2=a^2x^2+b^2y^2.$$

(*Lucknow*, 1967; *Rajasthan*, 1973; *Gorakhpur*, 1973)

Hint. See solved example 4 of § 10.7.

10. If the point of the intersection of the ellipse $x^2/a^2+y^2/b^2=1$ and $x^2\alpha^2+y^2/\beta^2=1$ be the end points of conjugate diameters of the former, prove that

$$\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}=2.$$

11. From a point P of an ellipse two tangents are drawn to the circle on the minor axis as diameter. Prove that these tangents will meet the diameter of right angles to CP in points lying on two fixed straight lines parallel to the major axis.

12. If the normals at four points (x_r, y_r), $r=1, 2, 3, 4$ on the ellipse $x^2/a^2+y^2/b^2=1$ are concurrent, prove that

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4.$$

(*U. P. C. S.*, 1971; *Gorakhpur*, 1976)

13. Show that the product of the lengths of perpendiculars from centre of an ellipse to a point of conjugate lines depends only upon their direction and cannot exceed $\frac{1}{2}(a^2-b^2)$.

14. Perpendiculars PM and PN are drawn from any point P on the equi-conjugate diameters of an ellipse. Prove that the perpendicular from P on its polar line bisects MN.

15. If CP and CD be a pair of conjugate semi-diameters of an ellipse and p, d the points on the auxiliary circle corresponding to P, D respectively, show that $\angle pCd$ is a right angle.

(*I. A. S.*, 1968)

16. Show that the rectangle under the perpendiculars drawn to the normal at a point P of an ellipse from the centre and from the pole of the normal is equal to the rectangle under the focal distances of P.

17. If λ, λ' be the angles which any two conjugate diameters of an ellipse subtend at any fixed point on it, prove that $\cot^2\lambda+\cot^2\lambda'$ is constant. (*Ranchi*, 1968)

18. A diameter PP' of the ellipse $x^2/a^2+y^2/b^2=1$ being taken, the normal at P' intersects the ordinate at P in Q. Prove that the locus of Q is the ellipse

$$\frac{x^2}{a^2}+\frac{b^2y^2}{(2a^2-b^2)^2}=1.$$

19. Show that the locus of the intersection of normals at the extremities of the conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$ is the curve

$$2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2. \qquad (\textit{Agra}, 1969)$$

20. CP and CD are conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$. Prove that the locus of the arthocentre of the traingle CPD is the curve

$$2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2.$$

Hint. See *Q.* 19 above.

21. Show that the locus of the foot of the perpendicular on a varying tangent to an ellipse from either of its foci is a cencentric circle. (*I. A. S.*, 1975)

22. The eccentric angles of two points P and Q on the ellipse $x^2/a^2+y^2/b^2=1$ are ϕ_1 and ϕ_2. Prove that the area of the parallelogram formed by the tangents at the ends of the diameters through P and Q is $4ab\operatorname{cosec}(\phi_1\sim\phi_2)$ and hence that it is least when P and Q are at the extremities of conjugate diameters.

23. Through a fixed point P a pair of lines is drawn parallel to a variable pair of conjugates diameters of a given ellipse. The lines meet the principal axes in Q and R respectively. Show that the middle point of QR lies on a fixed line.

24. If any pair of conjugate diameters of an ellipse ut cthe tangents at a point P in T_1 and T_2, show that

$$PT_1 . PT_2=CD^2$$

where CD is the diameter conjugate to CP. (*I. A. S.*, 1964)

Hint. Take CP and CD as coordinate axes. The lines $y=m_1x$, $y=m_2x$ will be conjugate diameters if

$$m_1m_2=-\frac{b'^2}{a'^2}, \text{ where } CP=a' \text{ and } CD=b'.$$

25. If through a given point on an ellipse any two lines at right angles to each other be drawn to meet the curve, show that the line joining their extremities will pass through a fixed point on the normal.

26. The tangent at any point P of an ellipse cuts the equi-conjugate diameter in T_1 and T_2. Show that the triangles T_1CP, T_2CP are in the ratio $CT_1^2 : CT_2^2$, where C is the centre of the ellipse.

27. If l_1, l_2 be the lenghts of two tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at right angles to one another, prove that

$$\frac{4(a^2+b^2)^3}{l_1^2+l_2^2}=\left\{a^2+b^2+a^2b^2\left(\frac{1}{l_1^2}+\frac{1}{l_2^2}\right)\right\}^2.$$

28. P, Q, R three points, 'α' 'β', 'γ' on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$. Show that the area of the triangle PQR is

$$2ab\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2},$$

and that of the triangle formed by the tangents at P, Q, R is

$$ab\tan\frac{\alpha-\beta}{2}\tan\frac{\beta-\gamma}{2}\tan\frac{\gamma-\alpha}{2} \qquad (\textit{I. A. S.}, 1976)$$

29. The orthocentre of the triangle, formed by the two tangents which can be drawn from a point to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

and the chord of contact, lies on the ellipse. Show that the locus of the point is the curve

$$\frac{x^2}{a^2}\{y^2(a^2-b^2)+b^2(a^2+b^2)\}^2$$
$$+\frac{y^2}{b^2}\{x^2(a^2-b^2)-a^2(a^2+b^2)\}^2=(a^2y^2+b^2x^2)^2.$$

3. Two conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet a fixed line $px+qy=1$ in P and Q and the lines through P and Q respectively perpendicular to these diameters meet in T. Prove that the locus of T is the line.

$$a^2px+b^2qy=a^2+b^2. \qquad (\textit{Andhra}, 1961)$$

31. A triangle circumscribes the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its centroid lies on the axis of x at a distance c from the centre; prove that the angular of the triangle lie on the conic.

$$\frac{(x-3c)^2}{a^2}+\frac{y^2(a^2-9c^2)}{a^2b^2}=4.$$

CHAPTER XI

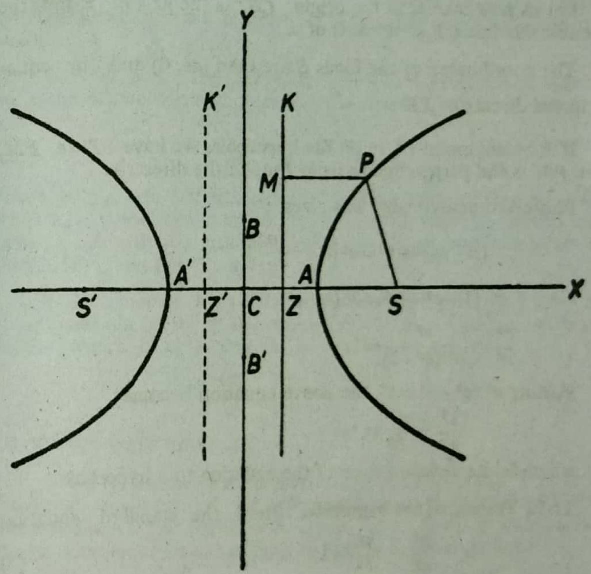
THE HYPERBOLA

11.1 Definition. A conic section in which the eccentricity is greater than unity is called a hyperbola.

Thus hyperbola is the locus of a point which moves such that its distance from a fixed point (the focus) is $e(>1)$ times its distance from a fixed straight line (the directrix).

The constant e is called the *eccentricity* of the hyperbola.

11. 11 Equation of the hyperbola.



Let S be the focus and ZK the directrix of a hyperbola. Draw SZ perpendicular to the directrix.

Since $e>1$, there exist two points A and A' which divide SZ internally and externally in the ratio $e:1$.

Let $AA'=2a$, and let C be the middle point of AA'.

From the definition, both A and A' lie on the hyperbola.

Now, $SA=e.AZ,$

and $SA'=e.ZA'.$

Therefore, $SA+SA'=e\,(AZ+ZA')$

or $2SC=2ae,$

i.e, $SC=ae.$

Further, $SA'-SA=e\,(ZA'-AZ),$

or $2a=e\,(AA'-2AZ).$

$=2e.\ CZ,$

i.e. $CZ=\frac{a}{e}.$

Let us now take C as the origin, CS as the axis of x and the perpendicular line CY as the axis of y.

The coordinates of the focus S are then $(ae, 0)$ and the equation to the directrix ZK is $x=\frac{a}{e}$.

If P be any point (x, y) on the hyperbola, we have $SP=e.\ PM$, where PM is the perpendicular from P upon the directrix.

Expressed analytically, this gives

$$(x-ae)^2+y^2=a^2\left(x-\frac{a}{e}\right)^2,$$

or $x^2\,(1-e^2)+y^2=a^2\,(1-e^2),$

i.e., $$\frac{x^2}{a^2}-\frac{y^2}{a^2\,(e^2-1)}=1.$$

Putting $a^2\,(e^2-1)=b^2$, the above equation becomes

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

which is the *standard form* of the equation to a hyperbola.

11.12 Tracing of the hyperbola. From the standard equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \cdots (1)$$

we see that the hyperbola is symmetrical about both coordinate axis. The centre is the origin itself.

Solving equation (1) for y,

$$y=\pm b\sqrt{\left(\frac{x^2}{a^2}-1\right)},$$

from which we see that corresponding to any value of x numerically greater than a, there are two equal and opposite values of y. The curve therefore does not lie between A and A'.

If $x=a$, $y=0$ and if $x\to\pm\infty$, y also tends to $\pm\infty$.

The curve thus consists of two infinite branches passing through A and A' respectively and lying one to the right of A, and the other to the left to A'.

If $x=0, y=\pm b\sqrt{(-1)}$. The y-axis thhrefore intersects the curve in imaginary points.

The line AA' is called the **transverse axis.** If we take points B, B' on y-axis, each distant b from the centre, BB' is called the **conjugate axis.**

As for the ellipse, there is a second focus S', $(-ae, 0)$ and a corresponding second directrix $Z'K'$ at the same distance from C as ZK.

Further, as for the ellipse, the locus of a point which moves in the plane of two perpendicular straight lines $u_1=0, u_2=0$ such that

$$\frac{p_1^2}{a^2}-\frac{p_2^2}{b^2}=1,$$

where p_1, p_2 are perpendicular distance of the moving point from $u_1=0$, $u_2=0$, and both a and b are real is a hyperbola whose transverse axis (length $2a$) lies along the line $u_2=0$ and conjugate axis (length $2b$) along the line $u_1=0$.

11.13 Position of a point relative to a hyperbola. As in the case of the parabola (§ 9.12), the point (x', y') lies within, upon, or with out the hyperbola according as

$$\frac{x'^2}{a^2}-\frac{y'^2}{b^2}-1$$

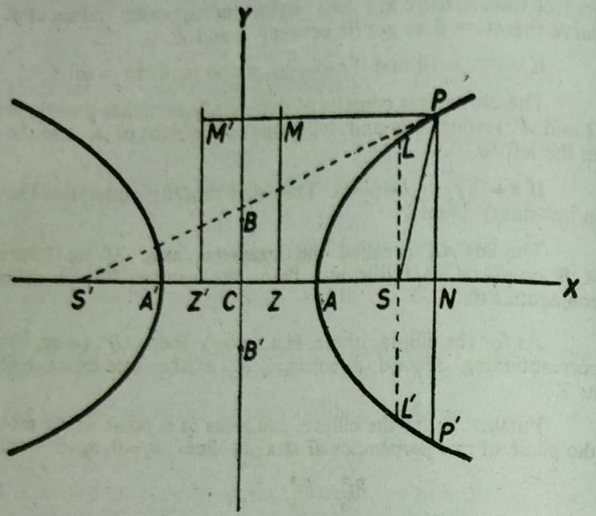
is positive, zero or negative.

11.14 A Geometrical property. The perpendicular PN to the transverse axisis called the **ordinate** of the point P and PNP' the **double ordinate.** The double ordinate through a focus is called a **latus rectum.** The length of either latus rectum is easily seen to be

$$\frac{2b^2}{a}.$$

If (x, y) be the point P, then fromt he equation to the hyperbola,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$



i.e.,
$$\frac{CN^2}{CA^2}-\frac{PN^2}{CB^2}=1,$$

or
$$\frac{PN^2}{CB^2}=\frac{CN^2}{CA^2}-1$$
$$=\frac{(CN+CA)\,(CN-CA)}{CA^2},$$
$$=\frac{NA'.\,AN}{CA^2},$$

where C is the centre, AA' the transverse axis and B an extremity of the minor axis of the hyperbola.

The equation to the hyperbola therefore expresses the geometrical property

$$PN^2 : AN.\,NA'=CB^2 : CA^2.$$

11.15 Focal distances of a point. In the figure of preceding article, if PM be the perpendicular from P upon the directrix ZM corresponding to the focus S,

$$PS=e.PM=e.NZ$$
$$=e\,(CN-CZ)=ex-a,$$

Similarly, if PM' be the perpendicular from P upon the directrix $Z'M'$ corresponding the second focus S',

$$PS'=e.PM'$$
$$=e.\,(CN+CZ')$$
$$=ex+a.$$

Therefore, $PS'-PS=2a$.

We thus conclude that the difference of the focal distances of a point on a hyperbola is constant and equal to the transverse axis.

11.16 Rectengular hyperbola. A hyperbola for which $a=b$ is said to be *rectangular* or *equilateral*. As we shall see later, the asymptotes of such a hyperbola arc at right angles to each other.

We have $$b^2=a^2\,(e^2-1).$$

For a rectangular hyperbola, this gives
$$e^2-1=1$$
i.e., $$e=\sqrt{2}.$$

The eccentricity of a rectangular hyperbola is therefore equal to $\sqrt{2}$.

11.17 Limiting case of a hyperbola. Writing the equation
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$
is
$$\frac{x^2}{a^2}-\frac{y^2}{a^2\,(e^2-1)}=1,$$
or
$$x^2-\frac{y^2}{e^2-1}=a^2,$$
we see that if we keep e fixed and let $a\to 0$, to the equation of the hyperbola becomes
$$x^2-\frac{y^2}{e^2-1}=0,$$
which represents a pair of straight lines.

A pair of straight lines is thus the limiting case of a hyperbola whose axes are infinitesimal, while their ratio is finite.

11.18 Coordinates of a point in terms of a single parameter. It is easy to see that the coordinates (x, y) of a point on the hyperbola
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$
can be expressed in terms of a single parameter θ as
$$x=a\sec\theta,\ y=b\tan\theta,$$

or, in terms of single parameter t, as

$$x=a \cosh t, \; y=b \sinh t,$$

Examples

1. Find the equation to the hyperbola whose directrix is $x+2y=1$, focus (2, 1) and eccentricity 2.

Ans. $x^2-16xy-11y^2-12x+6y+21=0$.

2. In a rectangular hyperbola, prove that

$SP.\,S'P=CP^2$. (*Ranchi*, 1972)

3. *Find the eccentricity of the hyperbola* $x^2-2x-4y^2=0$.

Solution. The given equation can be written as

$$(x-1)^2-4y^2=1,$$

or

$$\frac{(x-1)^2}{1}-\frac{y^2}{\frac{1}{4}}=1.$$

The transverse axis of the hyperbola is of length 2 and the conjugate axis is of length 1. The eccentricity is $\sqrt{1+\frac{1}{4}}=\frac{\sqrt{5}}{2}$.

4. The equations of the transverse and conjugate axes of a hyperbola are respectively $x+2y=3$, $2x-y+4=0$, and their respective lengths are $\sqrt{2}$ and $\frac{2}{\sqrt{3}}$. Find the equation of the hyperbola.

Ans. $x^2-4xy-2y^2+10x+4y=0$.

Hint. The equation to the hyperbola is

$$\frac{\left(\frac{2x-y+4}{\sqrt{5}}\right)^2}{\frac{1}{2}}-\frac{\left(\frac{x+2y-3}{\sqrt{5}}\right)^2}{\frac{1}{3}}=1.$$

5. On a level plane, the crack of the rifle and the thud of the ball striking the target are heard at the same instant. Show that locus of the bearer is a hyperbola.

11.2 Tangent and other loci.

Since the equation of the hyperbola differs from that of the ellipse in having $-b^2$ for b^2, several of the results obtained for the ellipse hold good for the hyperbola when the sign of b^2 is changed.

We shall give below a few results for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ which may be obtained independently as for the general conic in Chapter VI or deduced from those obtained for the ellipse.

(*i*) The tangent at any point (x', y') on the curve is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1,$$

which is also the equation to the chord of contact of the tangents from (x', y'), and the polar of (x', y').

(*ii*) The normal at any point (x', y') on the curve is

$$\frac{x-x'}{\frac{x'}{a^2}}+\frac{y-y'}{\frac{y'}{b^2}}=0.$$

(*iii*) The straight line $y=mx+\sqrt{a^2m^2-b^2}$ is a tangent to the curve for all values of m.

(*iv*) The straight line $x \cos \alpha+y \sin \alpha-p=0$ touches the curve if $p^2=a^2 \cos^2 \alpha-b^2 \sin^2 \alpha$.

(*v*) The equation of the chord whose middle point is (x', y') is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=\frac{x'^2}{b^2}-\frac{y'^2}{b^2}.$$

(*vi*) The diameters $y=mx$, $y=m'x$, are conjugate if

$$mm'=\frac{b^2}{a^2}.$$

(*vii*) The equation of the director circle is

$$x^2+y^2=a^2-b^2,$$

which is imaginary when $b>a$.

(*viii*) The equation of the pair of tangents from (x', y') is

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}-\frac{y'^2}{b^2}-1\right)=\left(\frac{xx'}{a^2}-\frac{yy'}{b^2}-1\right)^2.$$

11.21 Tangent and normal at the point ($a \sec \theta$, $b \tan \theta$).

The equation of the tangent at $(a \sec \theta, b \tan \theta)$ is

$$\frac{x}{a} \sec \theta-\frac{y}{b} \tan \theta=1,$$

i.e.,

$$\frac{x}{a}-\frac{y}{b} \sin \theta=\cos \theta.$$

The equation of the normal at this point is

$$\frac{\sec \theta}{a}(y-b \tan \theta)+\frac{\tan \theta}{b}(x-a \sec \theta)=0,$$

i.e.,

$$by \cot \theta+ax \cos \theta=a^2+b^2.$$

From this it can be seen that four normals can be drawn to a hyperbola from a point in its plane.

11.3 The Asymptotes. A straight line which touches a curve at infinity but does not lie wholly at infinity is called an *asymptotes* of the curve.

Let us write the equation of the hyperbola as

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=1.$$

The straight line $\frac{x}{a}+\frac{y}{b}=k$ meets the hyperbola in points whose ordinates are given by

$$k\left(k-\frac{2y}{b}\right)=1.$$

Since any straight line meets a conic in two points the coefficient of y^2 in the above equation being zero, one of the values of y is infinite. The other value is also infinite if $k=0$.

The straight line $\frac{x}{a}+\frac{y}{b}=0$ therefore meets the hyperbola in two coincident points at infinity and is an *asymptote* by definition.

The other asymptote will similarly be seen to be

$$\frac{x}{a}-\frac{y}{b}=0.$$

In the case of the rectangular hyperbola $x^2-y^2=a^2$, the asymptotes are $x=\pm y$. The angle between them is 90°.

It will also be seen that the asymptotes of a hyperbola are the pair of tangents drawn from its centre.

The asymptotes of an ellipse are imaginary.

11.4 The Conjugate hyperbola. The hyperbola which has for its transverse and conjugate axes the conjugate and transverse axes of another hyperbola is called the 'conjugate hyperbola'.

The hyperbola

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1 \qquad \text{...(1)}$$

is conjugate to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \qquad \text{...(2)}$$

The two have the same asymptotes, *viz.*,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0. \qquad \text{...(3)}$$

It should be observed that equation (3) differs from equation (2) by a constant, and that equation (1) differs from equation (3) by the same constant.

Any transformation of axes will change the left hand members of (1), (2) and (3) in exactly the same way and the right hand constants after the transformation will differ in the same manner as before.

Hence, the equation of the asymptotes differs from that of the hyperbola by a constant, and the equation of the conjugate hyperbola differs from that of the asymptotes by the same constant.

Examples

1. *Find the asymptotes of the hyperbola $x^2+3xy+2y^2+2x+3y=0$, and the equation to the conjugate hyperbola.*

Solution. Since the equation of the asymptotes differs from that of the hyperbola by a constant, let the asymptotes be

$$x^2+3xy+2y^2+2x+3y+c=0.$$

This represents a pair of straight lines. Hence,

$$2c+2.\tfrac{3}{2}.\ 1.\tfrac{3}{2}-1.\tfrac{9}{4}-2.1-c.\tfrac{9}{4}=0,$$

or

$$c=1.$$

The equation of the asymptotes is

$$x^2+3xy+2y^2+2x+3y+1=0,$$

and the equation of the conjugate hyperbola is

$$x^2+3xy+2y^2+2x+3y+2=0.$$

2. Prove that the asymptotes of the hyperbola

$$xy=hx+ky \text{ are } x=k,\ y=h.$$

3. *Find the asymptotes of*

$$x^2+xy-2y^2+2x+y+4=0,$$

and the general equation of all hyperbolas having the same asymptotes.

Solution. If the asymptotes are $x^2+xy-2y^2+2x+y+c=0$,

$$-2c+2.\tfrac{1}{2}\ 1.\tfrac{1}{2}-1.\tfrac{1}{4}+2.1-c.\tfrac{1}{4}=0,$$

from which $c=1$.

Hence, the equation of the asymptotes is

$$x^2+xy-2y^2+2x+y+1=0.$$

The general equation of all hyperbolas having the above asymptotes is

$$x^2+xy-2y^2+2x+y+\lambda=0,$$

where $\lambda\neq 1$.

4. Find the asymptotes of $xy-3x-2y=0$. What is the equation of the conjugate hyperbola ? (*Lucknow*, 1978; *Agra*, 1967)

Ans. $x=2, y=3$; $xy-3x-2y+12=0$.

5. Find the equation to the hyperbola which has $3x-4y+7=0$ and $4x+3y+1=0$ for its asymptotes and which passes through the origin. (*U. P. C. S.*, 1972; *Kanpur*, 1974)

Ans. $12x^2-7xy-12y^2+31x+17y=0$.

6. If e, e' be the eccentricities of a hyperbola and its conjugate, prove that

$$\frac{1}{e^2}+\frac{1}{e'^2}=1.$$ (*Gorakhpur*, 1968; *Jodhpur*, 1969)

7. Prove that the tangent to a hyperbola intercepted between the asymptotes is bisected at the point of contact.

11.5 Propositions on conjugate diameters. We shall prove the following propositions on conjugate diameters of hyperbola.

(1) *Diameters which are conjugate with respect to a hyperbola are also conjugate with respect to the conjugate hyperbola.*

Let the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The diameters $y=mx$, $y=m'x$ are conjugate with respect to the hyperbola if

$$mm'=\frac{b^2}{a^2}. \qquad ...(1)$$

Now, since the equation of the conjugate hyperbola differs from that of the hyperbola in having $-a^2$ and $-b^2$ in place of a^2 and b^2 respectively, the above diameters will be conjugate with respect to the conjugate hyperbola if,

$$mm'=\frac{-b^2}{-a^2}=\frac{b^2}{a^2}.$$

This is the same as relation (1) above. Hence the proposition is proved.

(2) *If a diameter meets a hyperbola in real points, it will meet the conjugate hyperbola in imaginary points; and the conjugate diameter will meet the conjugate hyperbola in real points.*

Let the equation to the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \qquad ...(1)$$

The equation of the conjugate hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1. \qquad ...(2)$$

Let us find the intersections of a diameter with the hyperbola and its conjugate. For this we convert equations (1) and (2) into polar coordinates. We thus have

$$\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}=\frac{1}{r^2} \qquad ...(3)$$

and

$$\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}=-\frac{1}{r^2}. \qquad ...(4)$$

If, for a given value of θ, $\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}$ is positive, the corresponding diameter has real intersections with (3) and imaginary intersections with (4).

A diameter of a hyperbola therefore does not intersect both the hyperbola and the conjugate hyperbola in real points. If it has real intersections with the hyperbola, its intersections with the conjugate hyperbola are imaginary.

Next, let $y=mx$ and $y=m'x$ be a pair of conjugate diameters of the hyperbola.

Then $$mm'=\frac{b^2}{a^2}. \qquad ...(5)$$

The abscissae of the points where the diameter $y=mx$ meets the hyperbola are given by

$$x^2=\frac{a^2b^2}{b^2-a^2m^2}.$$

For real intersections we have $b^2>a^2m^2$, or $m^2<\frac{b^2}{a^2}$. From equation (5) we have $m'^2>\frac{b^2}{a^2}$.

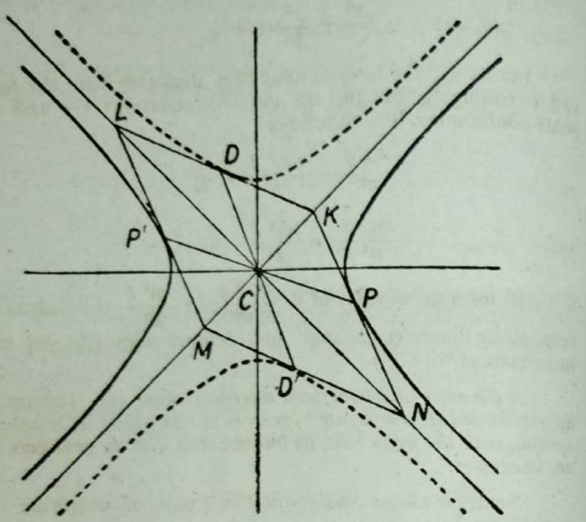
Now, the conjugate diameter $y=m'x$ meets the conjugate hyperbola in points whose abscissae are given by

$$x^2\left(\frac{1}{a^2}-\frac{m'^2}{b^2}\right)=-1, \text{ i.e., } x^2=\frac{a^2b^2}{a^2m'^2-b^2}.$$

Since $m'^2>\frac{b^2}{a^2}$, the intersections are real

(3) *If a pair of conjugate diameters meet the hyperbola and its conjugate in P and D, them* $CP^2-CD^2=a^2-b^2$.

(*Lucknow*, 1978; *Ranchi*, 1972)



Let P be the point $(a \sec \theta, b \tan \theta)$ on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The equation to CP is

$$y=\frac{b \sin \theta}{a}x.$$

Since the product of the slopes of CP and CD is $\frac{b^2}{a^2}$, the eqution to CD is

$$y=\frac{b}{a \sin \theta}x.$$

This meets the conjugate hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ in points $(a \tan \theta, b \sec \theta)$, and $(-a \tan \theta, -b \sec \theta)$.

The coordinates of D are therefore $(a \tan \theta, b \sec \theta)$.

Then $\quad CP^2=a^2 \sec^2 \theta+b^2 \tan^2 \theta,$

and $\quad CD^2=a^2 \tan^2 \theta+b^2 \sec^2 \theta.$

Subtracting, $CP-CD^2=a^2-b^2$.

(4) *The parallelogram formed by the tangents at the extremities of conjugate diameters has its vertices lying on the asymptotes and is of constant area.*

The tangent at P is

$$\frac{x}{a}-\frac{y}{b}\sin \theta=\cos \theta,$$

and the tangent at D to the conjugate hyperbola is

$$\frac{x}{a}\tan \theta-\frac{y}{b}\sec \theta=-1,$$

i.e.,

$$\frac{x}{a}\sin \theta-\frac{y}{b}=-\cos \theta.$$

The coordinates of K which is the point of intersection of these two tangents are given by

$$\frac{x}{a}=\frac{y}{b}=\frac{\cos \theta}{1-\sin \theta}.$$

K therefore lies on the asymptote $\frac{x}{a}=\frac{y}{b}$.

Similarly, the remaining angular points also lie on the asymptotes.

The area of the parallelogram formed by the tangents is four times the area of the parallelogram $CPKD$.

Now, $\quad CD=\sqrt{a^2 \tan^2 \theta+b^2 \sec^2 \theta},$

and the perpendicular 'p' from C upon the tangent at P is

$$\frac{\cos \theta}{\sqrt{\frac{1}{a^2}+\frac{\sin^2 \theta}{b^2}}}$$

$$=\frac{ab}{\sqrt{a^2 \tan^2 \theta+b^2 \sec^2 \theta}}$$

$\therefore \quad CD \cdot p=ab.$

The area of the parallelogram $KLMN$ is therefore $4ab$.

11.51 The equation of a hyperbola referred to any pair of conjugate diameters as axis.

The equation of the hyperbola referred to its transverse and conjugate axes is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

If the coordinate axes are rotated such that they coincide with a pair of conjugate diameters, the equation has the form (see Chapter IV)

$$Ax^2+2Hxy+By^2=1. \qquad ...(1)$$

Since all chords parallel to one diameter are bisected by the other, $H=0$. This simplifies (1) to

$$Ax^2+By^2=1. \qquad ...(2)$$

One of the semi-conjugate diameters is real, and the other imaginary. If a' and $\sqrt{-1}\,b'$ be their lengths, the point $(a', 0)$, $(0, \sqrt{-1}\,b')$ lie on (2).

Substituting, we get $A=\dfrac{1}{a'^2}$, $B=-\dfrac{1}{b'^2}$

Hence, the required equation is

$$\frac{x^2}{a'^2}-\frac{y^2}{b'^2}=1.$$

Examples

1. If the polars of (x_1, y_1), (x_2, y_2) with respect to the hyperbola $x^2/a^2-y^2/b^2=1$ are at right angles, show that

$$\frac{x_1x_2}{y_1y_2}+\frac{a^4}{b^4}=0. \qquad (Magadh, 1974)$$

2. Prove that the polar of any point on an asymptote of a hyperbola with respect to the hyperbola is parallel to that asymptote.

3. *Show that the locus of the middle points of normal chords of the rectangular hyperbola $x^2-y^2=a^2$ is*

$$(y^2-x^2)^3=4a^2x^2y^2.$$

(*Lucknow*, 1968; *U. P. C. S.*, 1969; *Kashmir*, 1973)

Solution. The equation of the normal to the rectangular hyperbola is

$$x\cos\theta+y\cot\theta=2a. \qquad ...(1)$$

If (h, k) be the middle point of this normal, the equation of the normal is also

$$xh-yk=h^2-k^2. \qquad ...(2)$$

Comparing coefficients in (1) and (2), we have

$$\cos\theta=\frac{2ah}{h^2-k^2} \text{ and } \sin\theta=-\frac{h}{k}.$$

On eliminating θ, these give

$$4a^2h^2k^2+h^2(h^2-k^2)^2=k^2(h^2-k^2)^2.$$

Hence, the locus of (h, k) is

$$(y^2-x^2)^3=4a^2x^2y^2.$$

4. Prove that the locus of poles with respect to $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ of any tangent to the circle whose diameter is the line joining the foci, is the ellipse

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}. \qquad (Rajasthan, 1966)$$

5. Prove that the locus of poles of normal chords of the hyperbola $x^2/a^2-y^2/b^2=1$ is

$$\frac{a^6}{x^2}-\frac{b^6}{y^2}=(a^2+b^2)^2. \qquad (Kashmir, 1974)$$

6. A straight line is drawn parallel to the conjugate axis of the hyperbola and meets it and the conjugate hyperbola in the points P and Q. Show that the tangent at P and Q meet on the curve

$$\frac{y^4}{b^4}\left(\frac{y^2}{b^2}-\frac{x^2}{a^2}\right)=\frac{4x^2}{a^2},$$

and the normals meet on the axis of x.

7. A circle cuts two fixed perpendicular lines so that each intercept is of given length. Prove that the locus of the centre of the circle is a rectangular hyperbola.

8. Find the locus of a point such that the angle between the tangents from it to a hyperbola is a equal to the angles between the asymptotes of the hyperbola.

$$\textit{Ans. } (a^2-b^2)^2\left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=(a^2-b^2-x^2-y^2)^2.$$

9. Show that the locus of the middle points of chords of constant length $2d$ of the rectangular hyperbola $xy=c^2$ is the curve

$$(x^2+y^2)(xy-c^2)=d^2xy. \qquad (Andhra, 1961)$$

10. *A* chord of the hyperbola $x^2/a^2-y^2/b^2=1$ touches the circle $x^2+y^2=c^2$. Show that the middle point of the chord lies on the curve

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right) \qquad (Agra, 1966)$$

11. Prove that the locus of the middle points of chords of the hyperbola $x^2/a^2-y^2/b^2=1$ passing through the fixed point (h, k) is a hyperbola whose centre is $(h/2, k/2)$. (*Ranchi*, 1976)

12. A line tnrough the origin meets the circle $x^2+y^2=a^2$ at P and the hyperbola $x^2-y^2=a^2$ at Q. Prove that the locus of the point

of intersection of the tangent at P to the circle with the tangent at Q to the hyperbola is the curve

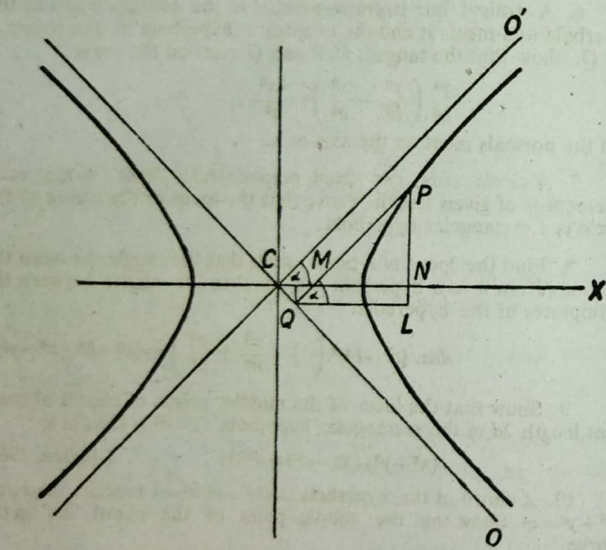
$$(a^2+4y^4)x^2=a^6.$$

13. A normal to the hyperbola $x^2/a^2-y^2/b^2=1$ meets the coordinate axis in M and N. Lines MP and NP are drawn perpendicular to the axes meeting at P. Prove that the locus of P is the hyperbola.

$$a^2x^2-b^2y^2=(a^2+b^2)^2. \qquad \textit{(Gorakhpur, 1969)}$$

14. Prove that the distance of any point from the centre of a rectangular hyperbola varies inversely as the perpendicular distance of its polar from the centre. (*Kashmir*, 1971)

11.6 Hyperbola when the asymptotes are coordinate axes.



Let the equation to the hyperbola referred to its transverse and conjugate axes be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \qquad ...(1)$$

We shall obtain the equation to this hyperbola when its asymptotes CO and CO' are taken as coordinate axes.

Let α be the angle which either asymptote makes with the transverse axis.

Then $$\tan\alpha=\frac{b}{a} \qquad ...(2)$$

Let (x, y) be the coordinates of any point P on the hyperbola (1) and let (x', y') be the coordinates of the same point when the equation to the hyperbola is referred to its asymptotes as coordinate axes.

If, then, PQ is drawn parallel to CO' to meet CO in Q,

$$CO=x',\ PQ=y'.$$

Draw QM perpendicular to CX, and let the ordinate PN be produced to meet the parallel to CX, through Q in L.

Now, $x=CN=CM+MN=x'\cos\alpha=y'\cos\alpha$

$$=\frac{(x'+y')\,a}{\sqrt{(a^2+b^2)}}, \text{ from (2).}$$

Similarly,

$$y=PN=PL-NL=PL-QM=y'\sin\alpha-x'\sin\alpha$$

$$=\frac{(y'-x')\,b}{\sqrt{(a^2+b^2)}}.$$

Substituting the above values x and y in (1),

$$\frac{(x'+y')^2}{a^2+b^2}-\frac{(y'-x')^2}{a^2+b^2}=1,$$

i.e., $$x'y'=\frac{a^2+b^2}{4}=c^2, \text{ say.}$$

Changing to current coordinates, the equation to the hyperbola when the asymptotes are taken as coordinate axes is

$$\mathbf{xy=c^2}.$$

Corollary. The equation to the conjugate hyperbola when the asymptotes are coordinate axes, is

$$xy=-c^2.$$

Note. If the axes are rectangular, the equation $xy=c^2$ represents a rectangular hyperbola whose asymptotes are the coordinate axes.

11.61 Equation of tangent to the hyperbola $xy=c^2$.

Let (x', y'), (x'', y'') be two points on the hyperbola $xy=c^2$.

Then $$x'y'=c^2 \qquad ...(1)$$

and $$x''y''=c^2. \qquad ...(2)$$

The equation to the chord joining the above points is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x'). \quad ...(3)$$

Now, from (1) and (2),

$$\frac{y''-y'}{x''-x'}=\frac{c^2\left(\frac{1}{x''}-\frac{1}{x'}\right)}{x''-x'}=-\frac{c^2}{x'x''}.$$

Hence equation (3) becomes

$$y-y'=-\frac{c^2}{x'x''}(x-x').$$

If the two points ultimately coincide, $x''\to x'$ and the chord becomes the tangent at (x', y').

The equation of the tangent a (x', y'), is therefore

$$y-y'=-\frac{c^2}{x'^2}(x-x')$$

$$=-\frac{y'}{x'}(x-x'), \text{ from (1).}$$

This can be written as

$$xy'+yx'=2x'y',$$

or,

$$\frac{x}{x'}+\frac{y}{y'}=2.$$

11.62 Coordinates in terms of a single parameter.

In terms of a signle parameter 't' the coordinates of any point on the hyperbola are easily seen to be $\left(ct, \frac{c}{t}\right)$.

The *equation of the tangent* at the point 't' is

$$\frac{x}{t}+ty=2c.$$

If the hyperbola is rectangular, the equation of the normal at $\left(ct, \frac{c}{t}\right)$ is

$$\frac{1}{t}\left(y-\frac{c}{t}\right)-t(x-ct)=0,$$

or

$$ty-t^3x+ct-c=0.$$

Examples

1. The chord PP' of the hyperbola meets the asymptotes in Q, Q'. Show that $QP=P'Q'$.

2. *Prove that the tangent at any point of a hyperbola cuts off a triangle of constant area from the asymptotes and that the portion of the tangent intercepted between the asymptotes is bisected at the point of contact.* (*Lucknow*, 1978)

Solution. Take the asymptotes of the hyperbola as coordinate axes. The equation of the hyperbola then is

$$xy=c^2.$$

If (x', y') be a point on this hyperbola, the equation of the tangent at this point is

$$\frac{x}{x'}+\frac{y}{y'}=2.$$

The intercepts which the tangent makes with the coordinate axes are $2x'$ and $2y'$.

The area of the triangle which the tangent cuts off from the asymptotes is

$$\tfrac{1}{2}.2x'.2y' \sin 2\alpha$$

$$=4x'y' \sin\alpha\cos\alpha$$

$$=\frac{4c^2ab}{a^2+b^2}=ab,$$

which is constant.

Further, the coordinates of the middle point of the portion of the tangent intercepted between the asymptotes are (x', y'). This proves the second part of the question.

3. If two sides of a triangle are given in position and the perimeter given in magnitude, prove that the middle point of the third side describes a hyperbola.

Hint. Take the two sides of the traingle as coordinate axes.

4. Show that $y=mx$ and $y=-mx$ are conjugate diameters of the hyperbola $xy=c^2$, whatever m may be.

5. *If a circle and the rectangular hyperbola* $xy=c^2$ *cut in the four points* $\left(ct_r, \frac{ct}{t_r}\right)$, $r=1, 2, 3, 4$, then show that $t_1t_2t_3t_4=1$.

Solution. The point $\left(ct, \frac{c}{t}\right)$ lies on the circle

$$x^2+y^2+2gx+2fy+c'=0,$$

Scanned by CamScanner

if $$c^2t^2+\frac{c^2}{t^2}\ 2gct+\frac{2fc}{t}+c'=0,$$

i.e., if $$e^2t^4+2gct^3+c't^2+2fct+c^2=0.$$

The roots of the above equation are t_1, t_2, t_3, t_4.

Hence, $$t_1t_2t_3t_4=\frac{c^2}{c^2}=1.$$

11.7 Miscellaneous solved examples.

1. *A parallelogram is constructed with its sides parallel to the asymptotes of a hyperbola, and one of its diagonals is a chord of the hyperbola; show that the othe diagonal passes through the centre.*

Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, and let the coordinates of the extremities of one diagonal be (x_1, y_1), (x_2, y_2). The equations of two adjacent sides of the parallelogram passing through (x_1, y_1), (x_2, y_2) respectively are

$$\frac{x}{a}-\frac{y}{b}=\frac{x_1}{a}-\frac{y_1}{b},$$

and $$\frac{x}{a}+\frac{y}{b}=\frac{x_2}{a}+\frac{y_2}{b}.$$

These intersect in the point

$$\left\{\frac{x_1+x_2}{2}-\frac{a}{2b}(y_1-y_2), \frac{y_1+y_2}{2}-\frac{b}{2a}(x_1-x_2)\right\}.$$

The other diagonal passes through this point and the middle point $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ of the first diagonal. Its equation is therefore

$$y-\frac{y_1+y_2}{2}=\frac{b^2(x_1-x_2)}{a^2(y_1-y_2)}\left(x-\frac{x_1+x_2}{2}\right),$$

i.e., $$a^2(y_1-y_2)\,y=b^2(x_1-x_2)\,x,$$

the constant term vanishing, since both (x_1, y_1), (x_2, y_2) lie on the hyperbola.

This obviously passes through (0, 0) which is the centre of the hyperbola.

2. *From two points (x_1, y_1), (x_2, y_2) are drawn tangents to the rectangular hyperbola $xy=c^2$; prove that the conic passing through the two points and through the four points of contact will be a cricle, if*

$$x_1y_2+x_2y_1=4c^2, \text{ and } x_1x_2=y_1y_2.$$

Solution. The equations to the two chords of contact are

$$xy_1+yx_1-2c^2=0 \qquad ...(1)$$

and $$xy_2+yx_2-2c^2=0. \qquad ...(2)$$

The equation of a conic passing through the points of intersection of the hyperbola with)1) and (2) is

$$(xy_1+yx_1-2c^2)\,(xy_2+yx_2-2c^2)+\lambda\,(xy-c^2)=0. \qquad ...(3)$$

This is a circle if the coefficient of xy in (3) is equal to zero and the coefficient of x^2 is equal to the coefficient of y^2. This gives

$$x_1y_2+x_2y_1+\lambda=0 \qquad ...(4)$$

and $$x_1x_2=y_1y_2 \qquad ...(5)$$

Further, since (3) passes through both (x_1, y_1) and (x_2, y_2), we have

$$2\,(x_1y_1-c^2)\,(x_1y_2+x_2y_1-2c^2)+\lambda\,(x_1y_1-c^2)=0,$$

and $$2\,(x_2y_2-c^2)\,(x_1y_2+x_2y_1-2c^2)+\lambda\,(x_2y_2-c^2)=0.$$

Since $x_1y_1-c^2\neq0$, and $x_2y_2-c^2\neq0$, we have from the above relations

$$2\,(x_1y_2+x_2y_1-2c^2)+\lambda=0. \qquad ...(6)$$

From (4) and (6),

$$x_1y_2+x_2y_1=4c^2.$$

3. *If t is the point $\left(ct, \frac{c}{t}\right)$ on the rectangular hyperbola $xy=c^2$, find the coordinates of the orthocentre of the triangle formed by the points t_1, t_2, t_3 and show that it lies on the rectangular hyperbola.*

(*I. A. S.*, 1971; *Allahabad*, 1966)

Solution. Let the coordinates of three points L, M, N on the hyperbola be

$$\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right), \left(ct_3, \frac{c}{t_3}\right)$$

The equation to LM is

$$y-\frac{c}{t_1}=\frac{\frac{c}{t_2}-\frac{c}{t_1}}{(ct_2-ct_1)}\,(x-ct_1),$$

i.e., $$x+t_1t_2y=c\,(t_1+t_2).$$

The equation of the perpendicular from N on this is

$$y-\frac{c}{t_3}-t_1t_2\,(x-ct_3)=0,$$

i.e., $$y+ct_1t_2t_3=t_1t_2\left(x+\frac{c}{t_1t_2t_3}\right). \qquad ...(1)$$

Similarly, the equation of the perpendicular from L upon MN is

$$y+ct_1t_2t_3=t_2t_3\left(x+\frac{c}{t_1t_2t_3}\right). \qquad \ldots(2)$$

The orthocentre of the triangle LMN, which is the point of intersection of (1) and (2), is

$$\left(-\frac{c}{t_1t_2t_3},\ -ct_1t_2t_3\right).$$

This obviously lies on the hyperbola.

4. *A circle with fixed centre $(3h, 3k)$ and of variable radius cuts the rectangular hyperbola $x^2-y^2=9a^2$ at the points A, B, C, D; prove that the locus of the centroid of the triangle ABC is given by*

$$(x-2h)^2-(y-2k)^2=a^2. \qquad \textit{(I. A. S., 1964)}$$

Rotating the axes so that the asymptotes of the hyperbola become the coordinate axes, the equation to the hyperbola becomes

$$xy=\frac{9a^2}{2}=c^2.$$

Referred to new axes, let the equation to the circle be

$$x^2+y^2+2gx+2fy+c'=0.$$

Then,
$$-g=\frac{3h-3k}{\sqrt{2}},\quad -f=\frac{3h+3k}{\sqrt{2}}.$$

If 't_1', 't_2', 't_3', 't_4' be the points A, B, C, D, then t_1, t_2, t_3, t_4 are the roots of the equation (Example 5 of § 11. 62)

$$c^2t^4+2gct^3+c't^2+2ftc+c^2=0.$$

Therefore,
$$t_1+t_2+t_3+t_4=-\frac{2g}{c} \qquad \ldots(1)$$

$$t_1t_2t_3+t_4\ (t_2t_3+t_3t_1+t_1t_2)=-\frac{2f}{c} \qquad \ldots(2)$$

and
$$t_1t_2t_3t_4=1. \qquad \ldots(3)$$

Now, if (x, y) be the centroid of the triangle ABC,

$$3x=c\ (t_1+t_2+t_3)$$
$$=-2g-ct_4 \qquad \ldots(4)$$

from (1); and
$$3y=\frac{c}{t_1}+\frac{c}{t_2}+\frac{c}{t_3}$$
$$=\frac{ct_4\ (t_2t_3+t_3t_1+t_1t_2)}{t_1t_2t_3t_4}$$
$$=-2f-\frac{c}{t_4} \text{ from (2) and (3)}, \qquad \ldots(5)$$

From (4) and (5)

$$(3x+2g)\ (3y+2f)=c^2.$$

Substituting for f and g from above and rotating the axes through 45° so that they coincide with the original axes, the required locus is

$$\left(3\frac{x-y}{\sqrt{2}}-3\ \frac{2h-2k}{\sqrt{2}}\right)\left(3\ \frac{x+y}{\sqrt{2}}-3\ \frac{2h+2k}{\sqrt{2}}\right)=c^2=\frac{9a^2}{2},$$

i.e.,
$$(x-2h)^2-(y-2k)^2=a^2.$$

5. *If O is the centre of the rectangular hyperbola through the four points A, B, C, D whose coordinates are $\left(t, \frac{1}{t}\right)$ where $t=a, b, c$ and if the perpendiculars from O to BC, AD meet AD, BC respectively in P, P', prove that if $abcd+1\neq 0$, the equation of the circle on PP' as diameter is*

$$(x+bcy-b-c)\ (x+ady-a-d)+(bcx-y)\ (adx-y)=0.$$

(Math. Tripos, 1962)

Solution. The equation to BC is

$$y-\frac{1}{b}=\frac{\frac{1}{c}-\frac{1}{b}}{c-b}\ (x-b),$$
$$=-\frac{1}{bc}\ (x-b),$$

i.e.,
$$x+bcy-b-c=0. \qquad \ldots(1)$$

Similarly, the equation to AD is

$$x+ady-a-d=0. \qquad \ldots(2)$$

The equations of the perpendiculars from O on (1) and (2) are respectively

$$bck-y=0, \qquad \ldots(3)$$

and
$$adx-y=0. \qquad \ldots(4)$$

Now, the circle on PP' as diameter passes through the feet of the perpendiculars from O on BC and AD, i.e., it passes through the four points of intersection of (1), (2) with (3) and (4).

The equation of a conic through these four points is

$$(x+bcy-b-c)\ (x+ady-a-d)+\lambda\ (bcx-y)\ (adx-y)=0.$$

Since this is a circle

$$(ad+bc)\ (1-\lambda)=0,\ i.e.,\ \lambda=1.$$

This gives the desired equation of the circle. If, however, $abcd+1=0$, the coefficients of x^2 and y^2 separately vanish and we merely get a straight line.

Examples on Chapter XI

1. If the normal to the rectangular hyperbola $xy=c^2$ at the point $(ct, c/t)$ on it intersects the hyperbola at $(ct', c/t')$, prove that $t^3t'=-1$. (*Punjab*, 1975)

2. Find the equation to the hyperbola conjugate to the hyperbola $(lx+my+n)\,(l'x+m'y+n')=k^2$.

Ans. $(lx+my+n)\,(l'x+m'y+n')+k^2=0$.

3. Prove that the chord which joins the points in which a pair of conjugate diameters meets the hyperbola and its conjugate is parallel to one asymptote and is bisected by the other.

4. The asymptotes of a hyperbola are $x+2y+3=0$ and $3x+4y+5=0$ and it passes through the point $(1, -1)$. Find its equation and the equation of the conjugate hyperbola. (*Punjab*, 1975)

Ans. $3x^2+10xy+8y^2+14x+22y+7=0$,
$3x^2+10xy+8y^2+14x+22y+23=0$.

5. Show that the coordinates of the point of intersection of two tangents to a hyperbola referred to its asymptote as axes are harmonic means between the coordinates of the point of contact.

6. Prove that the four equations

$$b\,(x\pm\sqrt{x^2-a^2})=a\,(y\pm\sqrt{y^2+b^2}),$$

represent respectively the portions of a hyperbola referred to its axes which lie in the four quadrants.

7. Show that if a chord PQ of a triangular hyperbola subtends a right angle at a point O on the curve, PQ is parallel to the normal at O. (*Gorakhpur*, 1964)

8. P and Q are two variable points on a rectangular hyperbola $xy=c^2$ such that the tangent at Q passes through the foot of the ordinate of P. Show that the locus of the point of intersection of tangent at P and Q is a hyperbola with the same asymptotes as the given hyperbola.

9. Show that the locus of the pole of any line with respect to a coaxal system of circle is a hyperbola, one of whose asymptotes is perpendicular to the given line, and the other is parallel to the radical axis of the system. (*Lucknow*, 1960)

10. Hyperbolas are drawn having a common transverse axis of length $2a$. On each is taken a point P such that its distance from the transverse axis is equal to its distance from an asymptote. Prove that the locus of P is a quartic curve $(x^2-y^2)^2=4x^2\,(x^2-a^2)$, referred to the common transverse axis and its perpendicular bisector as coordinate axes.

11. An ellipse and a hyperbola have the same principal axes. Show that the polar of any point on either curve with respect to the other, touches the first curve.

12. Find the equation of tangents common to the hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ and } \frac{y^2}{a^2}-\frac{x^2}{b^2}=1. \qquad (\textit{Magadh}, 1977)$$

Ans. $y\sqrt{2}=\pm\dfrac{x}{a}\sqrt{a^2+b^2}\pm\sqrt{a^2-b^2}$.

13. The chord of the hyperbola $x^2/a^2-y^2/b^2=1$ whose equation is $x\cos\alpha+y\sin\alpha=p$ subtends a right angle at its centre. Prove that it always touches a circle of radius $ab/\sqrt{b^2-a^2}$.

14. Prove that if the centre of the hyperbola whose equation is

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

be (α, β), then the equation of the asymptotes can be written in the form

$$ax^2+2hxy+by^2+2gx+2fy-g\alpha=f\beta=0.$$

Hint: The asymptotes pass through the centre of the conic.

15. A circle cuts the rectangular hyperbola $xy=1$ in points (x_r, y_r), $r=1, 2, 3, 4$. Prove that

$$x_1x_2x_3x_4=y_1y_2y_3y_4=1.$$

(*U. P. C. S.*, 1970; *Ranchi*, 1974)

16. The straight line $ax+by=1$ meets the hyperbola $xy=c^2$ in P and Q. Show that if C be the centre of the hyperbola the lines CP, CQ are perpendicular if

$$c^2e^2\,(a^2+b^2)-(2-e^2)\,(2c^2ab-1)=0,$$

where e is the eccentricity of the hyperbola.

17. If the normals at (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) on the rectangular hyperbola $xy=c^2$ meet in the point (α, β) prove that

$$\alpha=x_1+x_2+x_3+x_4,$$
$$\beta=y_1+y_2+y_3+y_4,$$

and $$x_1x_2x_3x_4=y_1y_2y_3y_4=-c^4.$$ (*Lucknow*, 1980)

18. A rectangular hyperbola whose centre is C is cut by any circle of radius r in the four points P, Q, R, S. Prove that

$$CP^2+CQ^2+CR^2+CS^2=4r^2.$$

(*I. A. S.*, 1973; *Allahabad*, 1960)

19. Show that if a rectangular hyperbola cuts a circle in four points, the centre of mean position of the four points is midway between the centres of the two curves. (*Delhi*, 1954)

20. The normals at four points on a rectangular hyperbola intersect in a point. Prove that the circle through any three of the first four points passes through the other extremity of the diameter through the fourth point.

Hint. The result follows from Examples Nos. 15 and 17.

21. Prove that in any rectangular hyperbola, the rectangle under the distances of any point of the curve from two fixed tangents is to the square on the distance from their chord of contact as $\cos\phi : 1$. where ϕ is the angle between the tangents.

22. At any point P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, a tangent is drawn to meet the asymptotes in Q and R. If C be the centre of the hyperbola, show that the locus of the centre of the circle circumscribing the triangle CQR is

$$4\,(a^2x^2-b^2y^2)=(a^2+b^2)^2. \qquad (Agra,\ 1958)$$

23. Show that an infinite number of triangles can be inscribed in the rectangular hyperbola $xy=c^2$ whose sides all touch the parabola $y^2=4ax$. (*Agra*, 1962; *Ranchi*, 1964)

Hint. The lines $x+t_1t_2y=c\,(t_1+t_2)$ touches the parabola if

$$m=-\frac{1}{t_1t_2} \text{ and } \frac{a}{m}=\frac{c\,(t_1+t_2)}{t_2t_2}.$$

Eliminating m, $a=\frac{c\,(t_1+t_2)}{-t_1^2t_2^2}$ which gives two values to t_2 corresponding to an arbitrary value of t_1.

24. With the point $(\alpha,\ \beta)$ as centre a family of circles is drawn to cut the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Prove that the locus of middle points of the chords of intersection is a rectangular hyperbola which passes through centre of the given conic.

Hint. The perpendicular from $(\alpha,\ \beta)$ to the chord

$$(x-x')\,(ax'+hy'+g)+(y-y')\,(hx'+by'+f)=0$$

passes through $(x',\ y')$.

25. An ellipse and a hyperbola are so related that the asymptotes of the hyperbola are conjugate diameters of the ellipse; prove that, by a proper choice of axes that their equations may be expressed in the forms

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=m.$$

26. Three tangents are drawn to the rectangular hyperbola $xy=a^2$ at the point (x_1, y_1), (x_2, y_2), (x_3, y_3) and form a triangle whose circumcircle passes through the centre of the hyperbola. Prove that

$$\frac{\Sigma x_1}{x_1x_2x_3}+\frac{\Sigma y_1}{y_1y_2y_3}=0,$$

and that the centre of the circle lies on the hyperbola.

27. If SY and $S'Y'$ be the perpendiculars from the foci S and S' on a tangent to the hyperbola at any point P, prove that Y and Y' lie on the circle with AA' as diameter, where A and A' are the vertices of the hyperbola. Show also that $SY.\ S'Y'$ is constant. (*I. A. S.*, 1968)

28. P, Q, R, S are four points on a rectangular hyperbola such that the chord PQ is perpendicular to the chord RS. Prove that each of the four points is the orthocentre of the triangle formed by the other three. (*I. A. S.*, 1978)

Hint. Let $\left(ct_r,\ \frac{c}{t_r}\right)r=1,2,3,\ 4$ be the coordinates of P, Q, R and S on the rectangular hyperbola $xy=c^2$. If PQ is perpendicular to RS,

$$t_1t_2t_3t_4=-1. \qquad ...(1)$$

The coordinates of the orthocentre of triangle PQR $\left(ct_r,\ \frac{c}{t_r}\right)$, $r=1,\ 2,\ 3$ are $\left(-\frac{c}{t_1t_2t_3},\ -ct_1t_2t_3\right)$, (Solved example 3 of § 11.7).

This is the same as $\left(ct_4,\ \frac{c}{t_4}\right)$. This proves the proposition.

CHAPTER XII

TRACING OF CONICS

12.1 Nature of a Conic. In the Chapter VI, we have seen that the equation to the conic is of the second degree. We shall now show that every cartesian equation of the second degree represents a conic.

Let $$ax^2+2hxy+by^2+2gx+2fy+c=0$$
be the general equation of the second degree referred to rectangular axes and let us turn axes through an angle $\frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$ without changing the origin.

With the above transformation of axes, the coefficient of xy in the equation of the conic becomes zero. Let the transformed equation be
$$a'X^2+b'Y^2+2g'X+2f'Y+c=0. \qquad \text{...(1)}$$
Since a rotation of axes without change of origin affects only the first and second degree terms in the equation of the conic, we see that in our case the expression $ax^2+2hxy+by^2$ has changed into $a'X^2+b'Y^2$. By invariants, therefore,
$$a+b=a'+b',$$
and
$$ab-h^2=a'b'. \qquad \text{...(2)}$$
If $ab-h^2=0$, then from (1) we have either $a'=0$ or $b'=0$.

When $a'=0$, equation (1) reduces to
$$b'Y^2+2g'X+2f'Y+c=0,$$
which is the equation of a parabola of which the axis is parallel to x-axis.

Similarly, when $b'=0$, equation (1) again represents a parabola of which the axis is now parallel to y-axis.

If $ab-h^2\neq 0$, then from (2) we conclude that both a' and b' are different from zero. Writing equation (1) as
$$a'\left(X+\frac{g'}{a'}\right)^2+b'\left(X+\frac{f'}{b'}\right)^2=\frac{g'^2}{a'}+\frac{f'^2}{b'}-c,$$
we conclude that in this case the conic is an ellipse or a hyperbola according as a' and b' have the same sign or different signs. If the signs are different and at the same time $a'+b'=0$, the hyperbola will be rectangular.

Now, when a' and b' have the same sign, $a'b'$ is positive, that is, from (2) $ab-h^2$ is positive.

When a' and b' have opposite signs, then from (2), $ab-h^2$ is negative.

When the axes are rectangular, the general equation
$$ax^2+2hxy+by^2+2gx+2fy+c=0$$
represents an ellipse or a hyperbola according as $ab-h^2$ is +ve or—ve. When $a+b=0$, the hyperbola is rectangular.

If $ab-h^2=0$, the conic is a parabola.

Note. In each case, it should be verified that the conic is not a pair of straight lines.

12.11 Nature of a conic when the axes are oblique. Let the equation of the conic be
$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \text{...(1)}$$
referred to oblique axes, the angle between the axes being ω.

Let us transform equation (1) to rectangular axis with the same origin and let it become
$$a'X^2+2hXY+b'Y^2+2g'X+2f'Y+c=0.$$

If the conic is not a pair of straight lines, we get from the preceding article that equation (2) represents

(*i*) a parabola if $a'b'-h'^2=0$,

(*ii*) an ellipse if $a'b'-h'^2>0$,

(*iii*) a hyperbola if $a'b'-h'^2<0$;

and (*iv*) a rectangular hyperbola if $a'+b'=0$.

Now, by invariants, the expressions
$$\frac{a+b-2h\cos\omega}{\sin^2\omega} \text{ and } \frac{ab-h^2}{\sin^2\omega}$$
retain the same values for a transformation of axes. In our case the angle for one set of axes is ω and for the other set of axes $90°$.

Hence,
$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=a'b'+b',$$
and
$$\frac{ab-h^2}{\sin^2\omega}=a'b'-h'^2.$$

Equation (1) thus represents a parabola, an ellipse or a hyperbola according as $ab-h^2$ is equal to, greater than or less than zero.

If $a+b-2h\cos\omega=0$, equation (1) reprssents a rectangular hyperbola.

12.12 Summary. We shall now give the summary of results for the conic represented by the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

The axes wil be rectangular unless stated otherwise.

The above equation represents

(*i*) a pair of straight lines if

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

The lines are parallel if, in addition, $ab-h^2=0$.

(*ii*) a circle if $a=b$ and $h=0$.

In the case of oblique axes the condition is

$$a:b:h=1:1:\cos\omega.$$

(*iii*) a parabola if

$$\Delta \neq 0, \text{ and } ab-h^2=0$$

The equation thus represents a parabola if the second degree terms form a perfect square.

(*iv*) an ellipse if $ab-h^2>0$ and the condition for a circle is not satisfied.

(*v*) a hyperbola if $ab-h^2<0$, $\Delta \neq 0$.

(*vi*) a rectangular hyperbola if $a+b=0$, $\Delta \neq 0$.

In the case of oblique axes, the condition is

$$a+b-2h\cos\omega=0. \Delta\neq 0.$$

12.2 Tracing of the parabola. We shall now find the axes and latus rectum of the parabola whose equation is

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \qquad \ldots(1)$$

Since equation (1) represents a parabola, $h^2=ab$, *i.e.*, the second degree terms form a perfect square.

We can therefore write equation (1) as

$\alpha^2x^2+2\alpha\beta xy+\beta^2y^2$

$$(\alpha x+\beta y)^2+2gx+2fy+c=0 \qquad \ldots(2)$$

where $\alpha^2=a$, $\beta^2=b$, $\alpha\beta=h$.

Now, we know that in a parabola the square of the distance of any point on the curve from the axis equal to its distance from, the tangent at the vertex multiplied by the length of the latus rectum.

Further, since the axes and the tangent at the vertex are perpendicular, we shall write equation (2) in a form which expresses the above property in relation to two perpendicular straight lines.

Now equation (2) can be written as

$$(\alpha x+\beta y+\lambda)^2=2x(\alpha\lambda+g)+2y(\beta\lambda-f)+\lambda^2-c.$$

We shall choose λ such that the lines

$$\alpha x+\beta y+\lambda=0, \qquad \ldots(3)$$

and

$$2x(\alpha\lambda-g)+2y(\beta\lambda-f)+\lambda^2-c=0 \qquad \ldots(4)$$

are perpendiculars.

The condition for this is

$$(\alpha\lambda-g)\alpha+(\beta\lambda-f)\beta=0,$$

which gives

$$\lambda=\frac{ag+\beta f}{\alpha^2+\beta^2}$$

With this value of λ, equation (3) represents the axes, equation (4) the tangent at the vertex and the point of intersection of (3) and (4) is the vertex of the parabola.

To find the latus rectum, let the equation be written as

$$\left(\frac{\alpha x+\beta y+\lambda}{\sqrt{\alpha^2+\beta^2}}\right)^2(\alpha^2+\beta^2)$$
$$=2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}\times\left\{\frac{(a\lambda-g)x+(\beta\lambda-f)y+\frac{1}{2}(\lambda^2+c)}{\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}}\right\}.$$

The length of the latus rectum is thus

$$\frac{2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}}{(\alpha^2+\beta^2)}.$$

Substituting for λ, the expression under the radical sign becomes

$$\frac{\beta^2(\alpha f-\beta g)^2+\alpha^2(\beta g-\alpha f)^2}{(\alpha^2+\beta^2)^2},$$

i.e.,

$$\frac{(\alpha f-\beta g)^2}{\alpha^2+\beta^2}.$$

The length of latus rectum is, therefore, the numerical value of

$$\frac{2(\alpha f-\beta g)}{(\alpha^2+\beta^2)^{3/2}}.$$

Example. Trace the parabola

$$16x^2-24xy-9y^2-104x-172y+44=0$$

and find the coordinates of its focus. (*Agra*, 1977; *I. A. S.*, 1974)

$(4x)^2-2\times(4x)(3y)+(3y)^2$

$(4x-3y)^2$

Solution. Let us write the given equation as

$$(4x-3y+\lambda)^2=x\,(104+8\lambda)+y\,(172-6\lambda)+\lambda^2-44.$$

Now choose λ such that the line

$$4x-3y+\lambda=0$$

and $$(104+8\lambda)\,x+(172-6\lambda)\,y+\lambda^2-44=0$$

are perpendiculars.

From the condition of perpendicularity, we get

$$\frac{4}{3}\times\left(-\frac{104+8\lambda}{172-6\lambda}\right)=-1. \text{ i.e., } \lambda=2.$$

The equation to the parabola now becomes

$$(4x-3y+2)^2=40\,(3x+4y-1)$$

which we can write as

$$\left(\frac{4x-3y+2}{5}\right)^2=8\left(\frac{3x+4y-1}{5}\right).$$

The equation of the axis is

$$4x-3y+2=0 \qquad \text{...(1)}$$

The equation of the tangent at the vertex is

$$3x+4y-1=0 \qquad \text{...(2)}$$

and the length of the latus rectum is 8.

The coordinates of the vertex which is the point of intersection of (1) and (2) are $(-1/5, 2/5)$.

To find the orientation of the parabola, we determine its intersection with the coordinate axes.

The parabola meets the x-axis in points whose distances from the origin are the roots of the equation

$$16x^2-104x+44=0$$

i.e., $$x=\frac{13\pm5\sqrt{5}}{4}.$$

The intersections with x-axis are thus both positive.

The distances from the origin of the points where the parabola meets the y-axis are the roots of the equation

$$9y^2-172y+44=0.$$

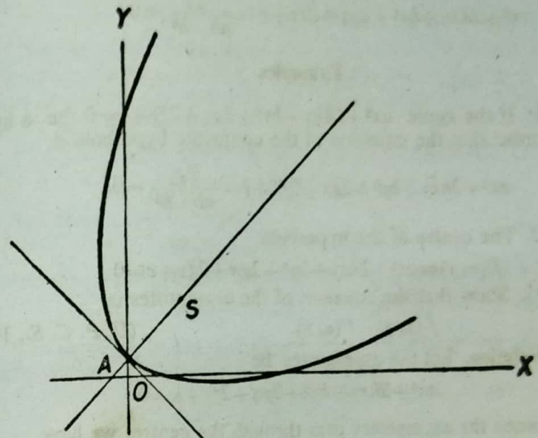
Both roots of this equation are again positive. The parabola thus intersects the y-axis also on the positive side of the origin. The shape of the curve is traced on the next page.

The focus S lies on the axis at a distance of 2 units from A. Its coordinates are therefore

$$(-\tfrac{1}{5}+2\cos\alpha,\ \tfrac{2}{5}+2\sin\alpha).$$

where $\tan\alpha=\frac{4}{3}$,

Substituting for $\cos\alpha$, $\sin\alpha$ the focus is the point (1, 2).



1.23 The Asymptotes. We shall now find the asymptotes of the conic represented by the general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

We know that the equation of the asymptotes differs from that of the conic only by a constant. The equation of the asymptotes of the given conic is therefore

$$ax^2+2hxy+by^2+2gx+2fy+c+k=0 \qquad \text{...(1)}$$

where k is so chosen that equation (1) represents a pair of straight lines.

Now, equation (1) represents a pair of straight lines if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c+k \end{vmatrix}=0,$$

i.e.. if $$k\,(ab-h^2)+\Delta=0,$$

where Δ is the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

The equation of the asymptotes is therefore

$$ax^2+2hxy+by^2+2gx+2fy+c-\frac{\Delta}{ab-h^2}=0.$$

Examples

1. If the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ be a hyperbola, prove that the equation of the conjugate hyperbola is

$$ax^2+2hxy+by^2+2gx+2fy+c-\frac{2\Delta}{ab-h^2}=0.$$

2. The centre of the hyperbola

$$f(x, y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

is (α, β). Show that the equation of the asymptotes is

$$f(x, y)=f(\alpha, \beta). \qquad (U.\ P.\ C.\ S.,\ 1976)$$

Solution. Let the asymptotes be

$$ax^2+2hxy+by^2+2gx+2fy+k=0.$$

Since the asymptotes pass through the centre, we have

$$a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+k=0,$$

i.e., $$-k=f(\alpha, \beta)-c.$$

Substituting the value of k in the equation of the asymptotes, we get the asymptotes as

$$f(x, y)=f(\alpha, \beta).$$

3. If the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

be a rectangular hyperbola, show that its equation referred to its asymptotes will be

$$2(h^2-ab)^{1/2}\,xy+k=0,$$

where k is a known constant.

Solution. Transferring the origin to the centre of the hyperbola, the axis remaining parallel to original axes, the equation to the hyperbola (see Chapter VI) becomes

$$ax^2+2hxy+by^2+\frac{\Delta}{ab-h^2}=0. \qquad ...(1)$$

Since the hyperbola is rectangular, the angle between the asymptotes is 90°. We now rotate the axes so that they coincide with the asymptotes. Equation (1) then transforms into

$$2\lambda xy+\frac{\Delta}{ab-h^2}=0. \qquad ...(2)$$

By invariants, we have from (1) and (2)

$$\lambda^2=h^2-ab.$$

The equation of the hyperbola referred to the asymptotes is therefore

$$2(h^2-ab)^{1/2}\,xy\pm\Delta=0.$$

The plus sign is taken when the hyperbola lies in the second and fourth quadrants, and the minus sign when the hyperbola lies in the first and third quadrants provided Δ is positive. If Δ is negative, the order of signs is reversed.

12.4 The lengths and positions of the axes of the conic $ax^2+2hxy+by^2=1$.

Let us consider the circle

$$x^2+y^2=r^2 \qquad ...(1)$$

of which the centre is at the centre of the conic

$$ax^2+2hxy+by^2=1. \qquad ...(2)$$

The equation to the pair of straight lines joining the origin to the intersections of (1) and (2) is

$$ax^2+2hxy+by^2=\frac{x^2+y^2}{r^2},$$

or $$\left(a-\frac{1}{r^2}\right)x^2+2hxy+\left(b-\frac{1}{r^2}\right)y^2=0. \qquad ...(3)$$

These lines will coincide in case they lie along either axis of conic, for the circle and the conic will then touch each other.

The length of a semi-axis of the conic is, therefore, that value of r for which (3) represents two coincident straight lines.

From this,

$$\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)=h^2.$$

If the roots of this quadratic in r^2 be r_1^2 and r_2^2, and if both r_1^2 and r_2^2 be positive, which will be the case when the conic is an ellipse, the lengths of the semi-axes will be r_1 and r_2, the greater value representing the semi-major axes. If the conic be a hyperbola, one root of the above quadratic will be positive, and the other negative. If r_1^2 is the positive root, length of the semi-transverse axis is r_1. The length of the semi-conjugate axis corresponding to the negative root r_2^2 is r_2.

When the left hand member of equation (3) is a perfect square, we can write it as

$$\left[\left(a-\frac{1}{r^2}\right)x+hy\right]^2=0.$$

Substituting for r^2, the equations of the axes are

$$\left(a-\frac{1}{r_1^2}\right)x+hy=0,$$

and $$\left(a-\frac{1}{r_2^2}\right)x+hy=0.$$

In the case of the ellipse, the equation of the major axis is the one corresponding to the greater of the values r_1^2, r_2^2. In the case of the hyperbola the equation of the transverse axis corresponds to the positive root of the quadratic in r^2.

Aliter. We can obtain the direction and magnitude of the axes of the general conic $ax^2+2hxy+by^2=1$ alternatively thus.

If the coordinate axes are rotated through an angle θ given by

$$\tan 2\theta=\frac{2h}{a-b}, \qquad \ldots(1)$$

the xy term in the equation of the conic disappears. The new axes, therefore, coincide with the principal axes of the conic.

From equation (1),

$$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2h}{a-b}=\frac{1}{k}, \text{ say}$$

$$\therefore \quad \tan^2\theta+2k\tan\theta-1=0. \qquad \ldots(2)$$

Let θ_1 and θ_2 be the two values of θ which satisfy (2).

Evidently, $\tan\theta_1 \tan\theta_2=-1$.

Therefore $\theta_1 \sim \theta_2=\frac{\pi}{2}$.

The equation of the conic in polar coordinates is

$$r^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)=1=\cos^2\theta+\sin^2\theta.$$

$$\therefore \quad r^2=\frac{1+\tan^2\theta}{a+2h\tan\theta+b\tan^2\theta}. \qquad \ldots(3)$$

Substituting in (3) the values of $\tan\theta$ obtained from (2), we get the length of axes.

The equations of the axes are

$$y=x\tan\theta_1 \text{ and } y=x\tan\theta_2.$$

Example. If the coordinate axes be inclined at an angle ω, show that the principal axes of lengths $2r_1$, $2r_2$ of the conic $ax^2+2hxy+by^2=1$ are

$$\left(a-\frac{1}{r_2^2}\right)x+\left(h-\frac{\cos\omega}{r_1^2}\right)y=0$$

and $$\left(a-\frac{1}{r_2^2}\right)x+\left(h-\frac{\cos\omega}{r_2^2}\right)y=0.$$

Hint. The equation of the circle is

$$x^2+2xy\cos\omega+y^2=r^2.$$

Proceed as in the fist method of the article.

12.5 Coordinates of the foci. Let us suppose that the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is an ellipse and that the lengths of the semi-major and semi-minor axes are r_1, r_2.

If e be the eccentricity,

$$e^2=1-\frac{r_2^2}{r_1^2}.$$

The foci are points on the major axis at a distance er_1 from the centre of the conic.

If θ is the angle of inclination of the major axis of the conic to the x-axis and (α, β) the centre of the conic, the coordinates of the foci are

$$(\alpha \pm er_1\cos\theta,\ \beta \pm er_1\sin\theta).$$

The same are the coordinates of the foci when the conic is a hyperbola, provided r is its semi-transverse axis.

12.6. Solved examples. We shall now illustrate the tracing of an ellipse and a hyperbola by considering the following examples :

Example 1. Trace the conic

$$22x^2-12xy+17y^2-112x+92y+178=0$$

and find the coordinates of its foci.

Solution. Since 17.22 -36 is positive, the conic is an ellipse.

The centre is obtained by solving

$$22x-6y-56=0$$
$$-6x+17y+46=0$$

as simultaneous equations (See Chapter VI).

Solving, the coordinates of the centre are $(2, -2)$. The equation of the ellipse when the origin is transferred to $(2, -2)$ is

$$22x^2-12xy+17y^2-56.2-46.2+178=0$$

i.e., $$22x^2-12xy+17y^2=26. \qquad \ldots(1)$$

Now the equation of the pair of straight lines joining the origin to the points where the circle $x^2+y^2=r^2$ cuts (1) is

$$22x^2-12xy+17y^2=\frac{26(x^2+y^2)}{r^2}$$

i.e., $$x^2\left(22-\frac{26}{r^2}\right)-12xy+y^2\left(17-\frac{26}{r^2}\right)=0. \qquad \ldots(2)$$

If r is a semi-axis,

$$\left(22-\frac{26}{r^2}\right)\left(17-\frac{26}{r^2}\right)=36$$

$A \times B = h^2$

or $$r^4-3r^2+2=0,$$

giving $$r^2=2 \text{ or } 1.$$

The length of the major axis is $2\sqrt{2}$ and its equation is $3x-2y=0$. The length of the major axis 2 and its equation is $2x+3y=0$.

Referred to original axes the equations of the major and minor axes are respectively

$$3(x-2)-2(y+2)=0,$$

i.e., $$3x-2y-10=0,$$

and $$2(x-2)+3(y+2)=0,$$

i.e., $$2x+3y+2=0.$$

Before we trace the curve, we shall find its intersection with the coordinate axes.

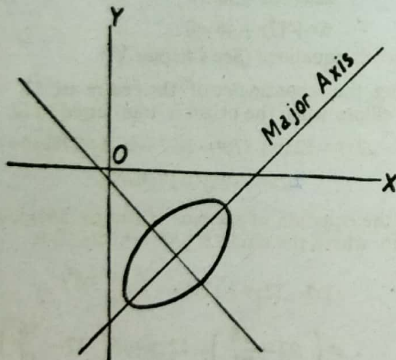
The intersections with x-axis are obtained by solving the equation $22x^2-112x+178=0$.

The roots of this equation being imaginary, the ellipse does not have real intersections with x-axis.

The intersections with y-axis are obtained by solving the equation

$$17y^2+92y+178=0.$$

The roots of this equation are also imaginary. The ellipse thus does not intersect the y-axis also in real points. The shape of the curve is as traced below :



The eccentricity of the ellipse is $\sqrt{1-\frac{1}{2}}=\frac{1}{\sqrt{2}}$.

The coordinates of the foci are therefore

$$\left(2\pm\frac{1}{\sqrt{2}}\cdot\sqrt{2}\cos\alpha,\quad -2\pm\frac{1}{\sqrt{2}}\cdot\sqrt{2}\sin\alpha\right),$$

where $\tan\alpha=3/2$.

Substituting for $\cos\alpha$ and $\sin\alpha$, the foci are the points

$$\left(2\pm\frac{2}{\sqrt{13}},\ -2\pm\frac{3}{\sqrt{13}}\right)$$

Example 2. Trace the conic

$$x^2-3xy+y^2+10x-10y+21=0.$$

Find its eccentricity and the equation of the asymptotes.

(*Kashmir*, 1972; *U. P. C. S.*, 1974; *Lucknow*, 1978)

Solution. Here $ab-h^2=1.1-\frac{9}{4}$, which is negative. Hence the equation represents a hyperbola.

The coordinates of the centre are obtained from the simultaneous equations

$$2x-3y+10=0,$$

$$-3x+2y-10=0.$$

Solving, the centre is the point $(-2, 2)$. Transferring the origin to $(-2, 2)$, the axes retaining their direction, the equation of the hyperbola becomes

$$x^2-3xy+y^2=-1. \qquad \ldots(1)$$

Now, the equation of the pair of straight lines joining the centre to the points where the circle $x^2+y^2=r^2$ cuts the hyperbola (1) is

$$x^2-3xy+y^2=-\frac{x^2+y^2}{r^2}$$

or $$x^2\left(1+\frac{1}{r^2}\right)3xy+y^2\left(1+\frac{1}{r^2}\right)=0.$$

If r is a semi-axis, $$\left(1+\frac{1}{r^2}\right)^2=\frac{9}{4},$$

giving $$r^2=2 \text{ or } -2/5.$$

The semi-transverse axis of the hyperbola is of length $\sqrt{2}$ and its equation is $x-y=0$. The semi-conjugate axis of the hyperbola is of length $\sqrt{2/5}$ and its equation is $x+y=0$.

The equations of the transverse and conjugate axes referred to old axes are $x-y+4=0$ and $x+y=0$ respectively.

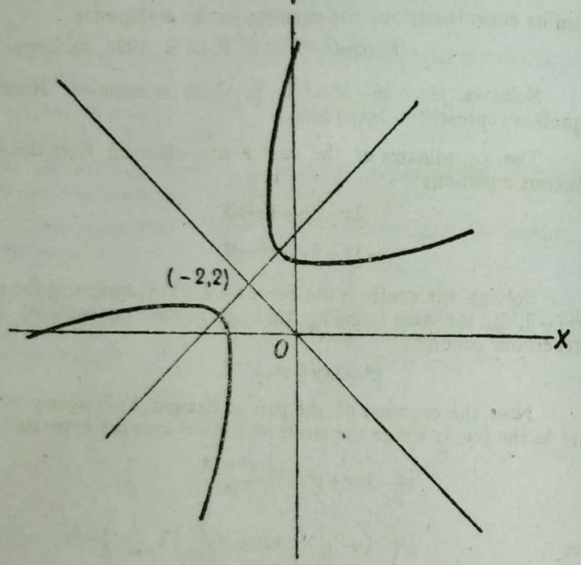
The intersection of the hyperbola with the x-axis are obtained from the equation $x^2+10x+21=0$, of which the roots are -3, -7. The intersection of the hyperbola with the y-axis are obtained from the equation $y^2-10y+21=0$, of which the roots are 3, 7.

The hyperbola, therefore, intersects the axes in the points $(-3, 0)$, $(-7, 0)$, $(0, 3)$ and $(0, 7)$. The shape of the curve is traced below.

If e be the eccentricity of the hyperbola, we have

$$\tfrac{2}{5}=2\,(e^2-1),\ \{b^2=a^2(e^2-1)\}$$

giving $\quad e=\sqrt{6/5}$.



Since the equation of the asymptotes differs from the equation of hyperbola, the same may be taken as

$$x^2-3xy+y^2+10x-10y+C_1=0 \qquad ...(2)$$

where C_1 is so chosen that (2) represents a pair of straight lines. This gives $C_1=20$. Consequently, the equation of the asymptotes is

$$x^2-3xy+y^2+10x-10y+20=0.$$

Example. 3. Show that the curve given by

$$x=\frac{at^2+bt^2+c}{\alpha t^2+\beta t+\gamma},\qquad y=\frac{a't^2+b't+c'}{\alpha t^2+\beta t+\gamma}$$

is a conic section. What is the condition that it may be a parabola ?

(*U. P. C. S.*, 1967)

Solution. The equation of the curve can be written as

$$t^2\,(a-\alpha x)+t\,(b-\beta x)+(c-\gamma x)=0$$
$$t^2\,(a'-\alpha y)+t\,(b'-\beta y)+(b'-\gamma y)=0.$$

Eliminating t, this equation is

$$x^2[(\alpha b'-\gamma a')^2-(\beta c'-\gamma b')(\alpha b'-\beta a')]+y^2[(a\gamma-c\alpha)^2-(b\gamma-\beta c)\,(a\beta-b\alpha)]$$
$$+xy[2\,(\alpha c'-\gamma a')\,(a\gamma-c\alpha)-(\beta c'-\gamma b')-(a\beta-b\alpha)-(b\gamma-c\beta)\,(\alpha b'-\beta a')]$$

+terms of lower degree in x and $y=0$.

Since this is an equation of second degree in x and y, the curve represented by the given equation is a conic.

Writing $A=(b\gamma-c\beta)$, $B=(c\alpha-a\gamma)$, $C=(a\beta-b\alpha)$

$A'\ (b'\gamma-c'\beta)$, $B'=(c'\alpha-a'\gamma)$, $C'=(a'\beta-b'\alpha)$.

The above equation can be written as

$$x^2\,(B'^2-A'C')+xy\ \ (2BB'-A'C-AC')+y^2\ \ (B^2-AC)+......=0.$$

This equation will represent a parabola when

$$(B^2-AC)\,(B'^2-A'C')=[BB'-\tfrac{1}{2}\,(A'C+AC')]^2$$

or $$(A'C-AC')^2=4\,(B'C-BC')\,(A'B-AB').$$

Putting the values of A, B, C, A', B' and C', the above equation

is $$\beta^2\,[\alpha\,(bc'-b'c)+\beta\,(ca'-c'a)+\gamma\,(ab'--a'b)]^2$$
$$=4\alpha\gamma\,[\,\alpha\,(bc'-b'c)+\beta\,(ca'-c'a)+\gamma(ab'-a'b)]^2$$

or $$(\beta^2-4\alpha\gamma)\,[\alpha\,(bc'-b'c)+\beta\,(ca'-c'a)+\gamma\,(ab'-a'b)]^2=0.$$

Hence the required condition is either $\beta^2=4\alpha\gamma$, or

$$\alpha\,(bc'-b'c)+\beta\,(ca'-c'a)+\gamma\,(ab'-a'b)=0.$$

Example 4. Trace the conic

$$16x^2-24xy+9y^2+77x-64y+95=0,$$

and find the coordinates of its focus.

(*Lucknow*, 1968; *Kashmir*, 1974)

Find also the length of the latus rectum and its equation.

Solution. In the equation of the conic, since $ab-h^2=0$, the conic is a parabola. The given equation can be written as

$$(4x-3y)^2=-77x+64y-95,$$

$$\text{or}\ \ (4x-3y+\lambda)^2=x\,(8\lambda-77)+y\,(64-6\lambda)+\lambda^2-95 \qquad ...(1)$$

where $$\left(\frac{4}{3}\right)\left(\frac{8\lambda-77}{64-6\lambda}\right)=1,\ \text{giving}\ \lambda=10,$$

The equation (1) becomes

$$(4x-3y+10)^2=3x+4y+5.$$

The equations of the axis of the parabola and tangent at vertex are

$$4x-3y+10=0 \text{ and } 3x+4y+5=0.$$

The coordinates of vertex are $\left(\frac{-11}{5}, \frac{2}{5}\right)$.

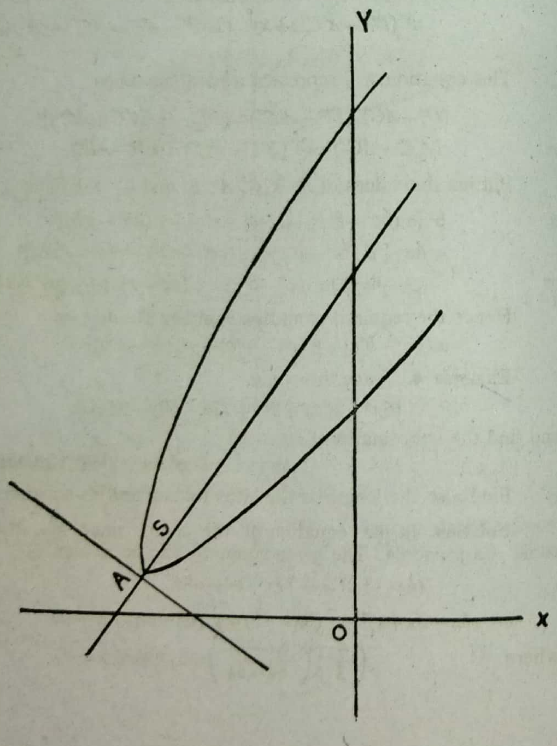
The equation (2) can also be written as

$$\left(\frac{4x-3y+10}{5}\right)^2=\left(\frac{1}{5}\right)\left(\frac{3x+4y+5}{5}\right).$$

Hence the length of latus rectum is 1/5.

Putting $y=0$ in the equation of the conic, we find x has imaginary values. The curve therefore does not intersect x-axis in real points. Putting $x=0$, we find $y=5$, 19/9.

The shape of the parabola is as given below.



If S be the focus of the parabola, then the distance $AS=1/20$. Hence the coordinates of the focus of the parabola are

$$\left(-\frac{11}{5}+\frac{1}{20}\cos\alpha, \frac{2}{5}+\frac{1}{20}\sin\alpha\right)$$

where $\tan\alpha=\frac{4}{3}$, or $\left(-\frac{217}{100}, \frac{11}{25}\right)$.

Further, the equation of the straight line parallel to $3x+4y+5=0$ and passing through the point $\left(-\frac{217}{100}, \frac{11}{25}\right)$ is $12x+16y+19=0$.

The equation of the latus rectum is therefore $12x+16y+19=0$.

Example on Chapter XII

1. Trace the following conics :

(i) $x^2-2xy+y^2-3x+y-2=0$. (*Kashmir*, 1975)

(ii) $x^2+2xy+y^2-2x-1=0$. (*Agra*, 1972)

(iii) $4x^2-4xy+y^2-8x-6y+5=0$. (*Kashmir*, 1973; *Gorakhpur*, 1975)

(iv) $11x^2+4xy+14y^2-26x-32y+23=0$. (*Lucknow*, 1965; *Agra*, 1975)

(v) $x^2+4xy+y^2-2x+2y+4=0$. (*Lucknow*, 1966; *Allahabad*, 1973)

(vi) $8x^2+4xy+5y^2=24(x+y)$.

(vii) $x^2-5xy+y^2+8x-20y+15=0$.

(viii) $x^2-10xy+y^2+2x-10y-11=0$. (*Lucknow*, 1980)

(ix) $5x^2-6xy+5y^2+22x-26y+29=0$.

(x) $4x^2+12xy+6y^2+9x+6y+2=0$.

2. Trace the conic

$9x^2-24xy+16y^2-18x-101y+19=0$. (*Lucknow*, 1978)

Find the coordinates of focus and the equation to the directrix. (*U. P. C. S.*, 1973)

3. Trace the conic

$36x^2+24xy+29y^2-72x+126y+81=0$. (*Kanpur*, 1973)

4. Trace the conic

$9x^2+24xy+16y^2-2x+14y+1=0$, and find the coordinates of its foci. (*Gorakhpur*, 1968)

5. Determine the lengths and position of the axes of the conic

$$17x^2+12xy+8y^2-46x-28y+33=0.$$

Find also the coordinates of its foci. (*Rohilkhand*, 1977)

6. Show that the conic

$$x^2-4xy+y^2-2x-20y-11=0$$

is a hyperbola. Find the coordinates of centre and show that the distance between the vertices of the two branches of the hyperbola is 12.

7. Show that $(-2, -1)$ is the centre of the conic

$$x^2+24xy-6y^2+28x+36y+16=0.$$

Trace the conic and find the equations of the asymptotes.

8. Trace the conic

$$17x^2-12xy+8y^2+46x-28y+17=0.$$

Find its eccentricity and the equations to the axes.

(*Agra*, 1973; *Gorakhpur*, 1976)

9. Sketch the conic

$$97x^2-60xy+72y^2-314x-348y+37=0.$$

(*Gorakhpur*, 1964)

10. Trace the conic

$$32x^2+52xy-7y^2-64x-52y-148=0,$$

and find the coordinates of its foci. (*Lucknow*, 1961)

11. Show that the conic $(a^2+b^2)(x^2+y^2)=(bx+ay-ab)^2$ is a parabola of latus rectum

$$\frac{2ab}{\sqrt{(a^2+b^2)}}.$$ (*Jodhpur*, 1969; *Kanpur*, 1973)

12. Draw a rough sketch of the conic

$$3(x^2+y^2)+2xy=4\sqrt{2}(x+y).$$

Determine the foci and show that the origin lies at an extremity of its principal axes. (*Lucknow*, 1960)

13. Find the equation to the directrix and coordinates of focus of the parabola

$$x^2-2xy+y^2-2x-2y+3=0.$$ (*I. A. S.*, 1961)

14. Trace carefully the conic represented by the equation

$$9x^2-6xy+y^2-14x-2y+12=0.$$

Find the coordinates of its focus and the equation of the directrix. (*Behrampur*, 1973)

15. Give a rough sketch of the conic

$$9x^2-24xy+16y^2-2x-39y-11=0.$$ (*I. A. S.*, 1969)

16. Trace carefully the conic

$$x^2-4xy-2y^2+10x+4y=0.$$ (*I. A. S.*, 1970)

17. Prove that the principal axes of the hyperbola

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

are parallel to the lines $h(x^2-y^2)=(a-b)xy$.

Find the principal axes of the conic

$$2x^2+12xy-7y^2-16x+2y-3=0.$$

Hint. The asymptotes are parallel to the lines $ax^2+2hxy+by^2=0$. Use the property that the axes bisect the angle between the asymptotes.

18. Show that the length of the semi-axis of the conic

$$ax^2+2hxy+by^2=1$$

are roots of the equation

$$(ab-h^2)r^4-(a+b)r^2+1=0.$$

Trace the conic

$$108x^2-312xy+17y^2+504x+522y-387=0.$$

19. The cartesian coordinates (x, y) of a point on a curve are given by $x=at^2+bt+c$, $y=\alpha t^2+\beta t+\gamma$. Show that the curve is a parabola of latus rectum

$$\frac{(a\beta-b\alpha)^2}{(a^2+\alpha^2)^{3/2}}.$$

20. Show that the lengths of semi-axes of the conic $ax^2+2hxy+by^2=d$ are $\sqrt{\{d/(a+h)\}}$ and $\sqrt{\{d/(a-h)\}}$ respectively and that their equations are $x^2-y^2=0$. (*Allahabad*, 1967; *Jodhpur*, 1968)

21. If r be the length of a semi-axis of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

show that

$$(ab-h^2)^3 r^4+\Delta(a+b)(ab-h^2) r^2+\Delta^2=0,$$

where

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$ (*I. A. S.*, 1976)

Hint. The equation of the conic when the origin is transferred to its centre is

$$ax^2+2hxy+by^2+\frac{\Delta}{ab-h^2}=0.$$

22. Trace the following curves :

(i) $|x|+|y|=1$.

(ii) $\frac{(x-1)^2}{4}+\frac{(y-2)^2}{9}=1$.

(iii) $(x+y)^2=4(x-y-1)$.

(iv) $xy-x-y=0$. (*I. A. S.*, 1977)

ANSWERS

1. (i) Parabola; axis : $x-y=1$; tangent at vertex : $x+y+3=0$; latus rectum : $(1/\sqrt{2})$.

(ii) Parabola; axis : $2x+2y=1$; tangent at vertex : $4x-4y+5=0$; latus rectum : $(1/\sqrt{2})$.

(iii) Parabola; axis : $2x-y=1$; tangent at vertex : $x+2y=1$; latus rectum : $(4/\sqrt{5})$.

(iv) Ellipse; centre : (1, 1); major axis : $x+2y=3$; minor axis : $2x-y=1$; lengths $2\sqrt{3/5}$ and $2\sqrt{3/5}$ respectively.

(v) Hyperbola; centre : $(-1, 1)$; transverse axis : $x+y=0$, length : $2\sqrt{6}$; conjugate axis : $y=x+2$, length: $2\sqrt{2}$.

(vi) Ellipse; centre : (1, 2); major axis : $2x+y=4$, length : 6; minor axis : $x-2y+3=0$, length : 4.

(vii) Hyperbola; centre : $(-4, 0)$; transverse axis : $x+y+4=0$, length : $2\sqrt{2/7}$; conjugate axis : $x-y+4=0$, length : $2\sqrt{2/3}$.

(viii) Hyperbola; centre : $(-1, 0)$; transverse axis : $x+y+1=0$, length : $2\sqrt{2}$; conjugate axis : $x-y+1=0$, length : $2\sqrt{3}$.

(ix) Ellipse; centre : $(-1, 2)$; major axis : $y=x+3$, length : 4; minor axis : $x+y=1$, length : 2.

(x) Two parallel straight lines.

2. Parabola; axis : $4y=3x+7$; tangent at vertex : $4x+3y+2=0$; latus rectum : 3.

3. Ellipse; centre : $(2,-3)$; major axis : $x-y+3=0$, length : 4; minor axis : $3x-4y=18$, length : 4.

4. Parabola; axis : $3x+4y+1=0$; tangeut at vertex : $4x-3y=0$; latus rectum : (2/5); focus : $\left(-\frac{1}{25}, -\frac{11}{50}\right)$.

5. Ellipse; major axis : $2x+y=3$; length $(4/\sqrt{5})$; minor axis : $2y=x+1$; length : $(2/\sqrt{5})$; foci : $\left(1\pm\frac{\sqrt{3}}{5}, 1\pm\frac{2\sqrt{2}}{5}\right)$.

6. $(-7,-4)$.

7. Transverse axis : $4y=3x+2$, length : $2\sqrt{3}$; conjugate axis : $4x+3y+11=0$, length : $2\sqrt{2}$; asymptotes given by

$$x^2+24xy-6y^2+28x+36y+46=0.$$

8. Ellipse; centre : $(-1, 1)$; major axis : $y=2x+3$; length : 4; minor axis : $x+2y=1$; length : 2; eccentricity: $\frac{\sqrt{3}}{2}$.

9. Ellipse; centre : $(1,-2)$; mojor axis : $3x-2y=7$; length 6; minor axis : $2x+3y+4=0$, length :4.

10. Hyperbola; centre : (1, 0) : transverse axis : $x-2y=1$; length 4; conjugate axis : $2x+y=2$; length : 6; and foci

$$\left(1\pm2\sqrt{\frac{13}{5}}, \pm\sqrt{\frac{13}{5}}\right).$$

12. Ellipse; centre : $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$; major axis : $x+y-\sqrt{2}=0$; length : $2\sqrt{2}$; minor axis : $x-y=0$; length : 2; foci $(0, \sqrt{2})$, $(\sqrt{2}, 0)$.

13. $x+y=1$; (1, 1).

14. Parabola; axis : $3x-y=2$; tangent at vertex : $x+3y=4$; latus rectum : $(2/\sqrt{10})$.

15. Parabola; axis : $4y=3x+3$; tangent at vertex : $4x+3y+4=0$; latus rectum : 1.

16. Hyperbola; centre : $(-1, 2)$; transverse axis : $x+2y=3$; length : $2/\sqrt{2}$; conjugate axis $2x-y=0$; length : $2/\sqrt{3}$.

17. Hyperbola; centre : (2, 3); transverse axis : $3y=5x+1$; length : 6; conjugate axis : $3x+4y=18$; length : 4.

18. Hyperbola; centre : (2, 3); transverse axis : $4x-3y+1=0$; length : 6; conjugate axis : $3x+4y-18=0$; length=4.

22. (i) The four sides of a square of length $\sqrt{2}$, the diagonals lying along the coordinate axes.

(ii) Ellipse; centre : (1, 2); major axis : $x=1$; length : 6; minor axis : $y=2$; length 4.

(iii) Parabola; axis : $x+y=0$; tangent at vertex : $y=x-1$; latus rectum : $2\sqrt{2}$.

(iv) Rectangular hyperbola; centre : (1, 1); asymptotes $x=1$, $y=1$.

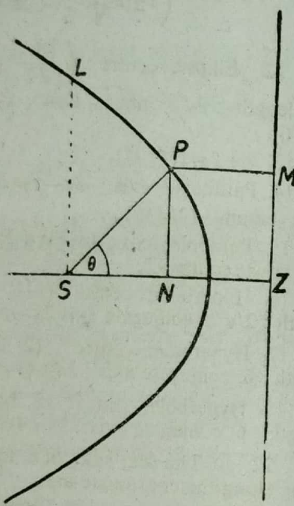
CHAPTER XIII

POLAR EQUATIONS

13.1 Polar equation of a conic when the focus is the pole.

We shall now obtain the equation of a conic in polar coordinates when the focus is at the pole.

Let S be the focus and ZM the directrix of the conic. Let SZ be perpendicular from S to the directrix, and let (r, θ) be the coordinates of a point P on the conic referred to S as pole and SZ as initial line.



From P draw PN and PM perpendiculars on SZ and ZM, and let e be the eccentricity of the conic and l its semi-latus rectum SL.

Then by the definition of the conic

$$SP=e.PM,$$

i.e.,

$$r=e.NZ$$
$$=e\,(SZ-SN)$$
$$=e\left(\frac{SL}{e}-SP\cos\theta\right)$$
$$=l-er\cos\theta,$$

or

$$\frac{l}{r}=1+e\cos\theta$$

which is the required polar equation.

Corollary 1. The equation of the conic when the axis SZ is inclined at an angle α to the initial line is

$$\frac{l}{r}=1+e\cos(\theta-\alpha),$$

for the angles between SP and SZ in this case is $\theta-\alpha$.

Corollary 2. The equation of the conic, when the positive direction of the initial line is SZ instead of SZ, is

$$\frac{l}{r}=1-e\cos\theta.$$

13.11 Directrices. We shall now find equations of the directrices of the conic $\frac{l}{r}=1+e\cos\theta$.

If (r, θ) be the coordinates of any point on the directrix ZM corresponding to the focus S, then

$$r\cos\theta=SZ=\frac{l}{e}.$$

The equation of the directrix corresponding to the focus which is the pole is, therefore

$$\frac{l}{r}=e\cos\theta.$$

To find the equation of the other directrix, let P' be a point (r, θ) on it and SZ' the perpendicular from S. Then,

$$SZ'=SP'\cos(\pi-\theta)=-r\cos\theta.$$

Now,

$$ZZ'=\frac{2a}{e} \text{ and } SZ=\frac{l}{e}.$$

Hence,

$$SZ'=ZZ'-SZ=\frac{2a}{e}-\frac{l}{e}$$
$$=\frac{2l}{e(1-e^2)}-\frac{l}{e}=\frac{l(1+e^2)}{e(1-e^2)},$$

since

$$l=\frac{b^2}{a}=a(1-e^2).$$

Equating the two values of SZ', we get

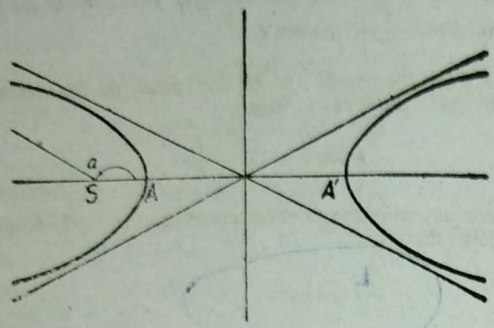
$$-r\cos\theta=\frac{l(1+e^2)}{e(1-e^2)},$$

or

$$\frac{l}{r}=-\frac{e(1-e^2)}{1+e^2}\cos\theta$$

as the equation of the other directrix.

13.12 Tracing the conic $\frac{l}{r}=1+e\cos\theta$. The student should find no difficulty in tracing the conic when $e=0$, 1 or <1. When $e>1$, that is, when the conic is a hyperbola, the equation $\frac{l}{r}=1+e\cos\theta$ represents only points on the branch nearer the pole if the radius vector is positive. Points on the further branch are obtained by taking negative values of the radius vector.



As θ increases from 0 to $\frac{\pi}{2}$, r increases from $\frac{l}{1+e}$ to l. As θ increases still further, $\cos\theta$ becomes negative and r goes on increasing. For $\cos\theta=-\frac{1}{e}$, r becomes infinite. A further increase in the value of θ, howsoever small, will make r negative. Hereafter r continues to be negative until $\theta=\pi$, when $r=-\frac{l}{e-1}$, which corresponds to A' on the further branch. If θ increases beyond π, r remains negative until θ appoaches $\left\{2\pi-\cos^{-1}\left(-\frac{1}{e}\right)\right\}$ where $r\to-\infty$. So that for values of θ between α and $2\pi-\alpha$, when $\alpha=\cos^{-1}\left(-\frac{1}{e}\right)$, the branch of the hyperbola farther from the pole is traced. For θ increasing from $2\pi-\alpha$ to 2π, r maintains a positive value, and points on the lower half of the branch nearer the pole are obtained.

The points at infinity on the curve are given by

$$\cos\theta=-\frac{1}{e}.$$

13.13 Asymptotes. It is obvious from the preceding article that the directions on the asymptotes of the hyperbola

$$\frac{l}{r}=1+e\cos\theta$$

are given by

$$\cos\theta=-\frac{1}{e}.$$

We know further that the asymptotes pass through the centre of the hyperbola.

Now the distance of the centre from the focus is ae, where a is the semi-transverse axis of the hyperbola.

The length of perpendicular from S upon either asymptote is

$$ae\sin\alpha=a\sqrt{e^2-1},$$

where

$$\cos\alpha=-\frac{1}{e}.$$

The angle which this perpendicular makes with the initial line is $-\left(\frac{\pi}{2}-\alpha\right)$, or $\left(\frac{\pi}{2}-\alpha\right)$ depending upon which asymptote is taken.

$$a\sqrt{e^2-1}=r\cos\left(\theta-\alpha+\frac{\pi}{2}\right),$$

and

$$a\sqrt{e^2-1}=r\cos\left(\theta+\alpha-\frac{\pi}{2}\right).$$

These can be written as

$$\frac{l}{r}=-\sqrt{e^2-1}\sin(\theta-\alpha),$$

and

$$\frac{l}{r}=\sqrt{e^2-1}\sin(\theta+\alpha).$$

This is, the asymptotes of the conic are the straight lines

$$\frac{1}{r}=\frac{\sqrt{e^2-1}}{e}(\sqrt{e^2-1}\cos\theta\pm\sin\theta)$$

Examples

1. Show that the equation $l/r=1+e\cos\theta$ and $l/r=-1+e\cos\theta$ represent the same conic.

(*Lucknow*, 1966; *Jiwaji*, 1967; *Allahabad*, 1971)

Hint. (r, θ) and $(-r, \theta+\pi)$ are the coordinates of the same point.

2. Prove that the equations $l/r=1-e\cos\theta$ and $l/r=-1-e\cos\theta$ represent the conic. (*Agra*, 1971)

3. PSP' is the focal chord of a conic. Prove that

$$\frac{1}{SP}+\frac{1}{SP'}=\frac{2}{l},$$

where l is the semi-latus rectum. (*Gorakhpur*, 1972; *Punjab*, 1976)

Solution. Let the equation of conic be

$$\frac{l}{r}=1+e\cos\theta$$

and let the chord PSP' make an angle α with the initial line. Then the vectorial angles of P and P' are respectively α and $\alpha+\pi$. From the equation of the conic

$$\frac{l}{SP}=1+e\cos\alpha \text{ and } \frac{l}{SP'}=1+e\cos(\alpha+\pi).$$

Adding these,
$$\frac{l}{SP}+\frac{l}{SP'}=2$$

or
$$\frac{1}{SP}+\frac{1}{SP'}=\frac{2}{l}.$$

3. If PSQ and PHR be two chords of an ellipse through the foci S and H, show that

$$\frac{PS}{SQ}+\frac{PH}{HR}$$

is independent of the position of P. (*Rajasthan*, 1966; *Agra*, 1974)

Hint. Use the result of the preceding example.

4. PSP' and QSQ' are two perpendicular focal chords of a conic; prove that

$$\frac{1}{SP.SP'}+\frac{1}{SQ.SQ'} \text{ is constant.}$$

(*Lucknow*, 1966; *Rajasthan*, 1972)

5. A circle passing through the focus of a conic whose latus rectum is $2l$ meets the conic in four points whose distances from the foci are r_1, r_2, r_3 and r_4. Show that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=\frac{2}{l}.$$

(*Allahabad*, 1971; *Rajasthan*, 1973; *U. P. C. S.*, 1974)

Solution. Take the focus as pole and the axis of the conic as the initial line. The equation of the conic can now be taken as

$$\frac{l}{r}=1+e\cos\theta. \qquad \text{...(1)}$$

The equation of the circle passing through the pole may be taken as

$$r=a\cos(\theta-\alpha) \qquad \text{...(2)}$$

where a is the diameter and α the angle which the diameter makes with the initial line. Eliminating θ between (1) and (2),

$$\left\{r-\frac{a\cos\alpha}{e}\left(\frac{l}{r}-1\right)\right\}^2=a^2\sin^2\alpha\left\{1-\frac{(l-r)^2}{e^2r^2}\right\}$$

or
$$e^2r^4+2r^3ae\cos\alpha+r^2(a^2-2ael\cos\alpha-a^2e^2\sin^2\alpha)$$
$$-2a^2lr+a^2l^2=0. \qquad \text{...(3)}$$

If r_1, r_2, r_3 and r_4 be the distances of the point of intersection from the focus, then these are the roots of equation (3).

Hence
$$r_1r_2r_3+r_1r_3r_4+r_1r_2r_4+r_2r_3r_4=\frac{2a^2l}{e^2} \qquad \text{...(4)}$$

and
$$r_1r_2r_3r_4=\frac{a^2l^2}{e^2}. \qquad \text{...(5)}$$

Dividing (4) by (5),

$$\frac{1}{r_4}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_1}=\frac{2}{l},$$

which proves the proposition.

6. A point moves so that the sum of its distances from two fixed points S, S' is constant and equal to $2a$. Show that P lies on the conic

$$\frac{a(1-e^2)}{r}=1-e\cos\theta,$$

referred to S as pole and SS' as the initial line, the SS' being equal to ae. (*Kanpur*, 1973)

Solution. Let the coordinates of P referred to S as pole and SS' as the initial line be (r, θ). Then, since $SP=r$, $S'P=2a-r$. From triangle SPS', we have

$$S'P^2=SP^2+SS'-2SP.SS'\cos\theta$$

or
$$(2a-r)^2=r^2+(2ae)^2-2.r.2ae\cos\theta$$

or
$$a-r=ae^2-er\cos\theta$$

giving
$$\frac{a(1-e^2)}{r}=1-e\cos\theta.$$

7. In any conic prove that the sum of reciprocals of two perpendicular focal chords is constant (*Jodhpur*, 1969; *Punjab*, 1976)

8. Show that the middle points of focal chords of a conic lie on another conic of the same kind. (*Lucknow*, 1963)

9. A circle of given radius passing through the focus S of a given conic intersects it in the points A, B, C and D. Show that

$$SA.SB.SC.SD \text{ is constant.}$$

13.2 Equation of the chord when the vectorial angles of the extremities are given.

Let the conic be $\frac{l}{r}=1+e\cos\theta, \qquad \text{...(1)}$

and let the vectorial angles of the extremities of the chord be $\alpha-\beta$, $\alpha+\beta$.

Now, the general equation of a straight line can be put in the form

$$\frac{l}{r}=A\cos(\theta-\alpha)+B\cos\theta \qquad ...(2)$$

as can easily be seen by converting equation (2) in cartesian coordinates.

Let (2) be the equation of the given chord. Then it must pass through points on (1), whose vectorial angles are $\alpha-\beta$ and $\alpha+\beta$.

Putting $\theta=\alpha-\beta$ and $\theta=\alpha+\beta$ in (1) and (2), and equating the values of r thus obtained, we get

$$1+e\cos(\alpha-\beta)=A\cos\beta+B\cos(\alpha-\beta),$$

and $$1+e\cos(\alpha+\beta)=A\cos\beta+B\cos(\alpha+\beta).$$

From these, $A=\sec\beta$, $B=e$.

Substituting the values of A and B in (2), the required equation of the chord is

$$\frac{l}{r}=\sec\beta\cos(\theta-\alpha)+e\cos\theta.$$

Corollary. The equation of the chord of the conic $\frac{l}{r}=1+e\cos(\theta-\gamma)$ joining the points whose vectorial angles are $\alpha-\beta$, $\alpha+\beta$ is

$$\frac{l}{r}=\sec\beta\cos(\theta-\alpha)+e\cos(\theta-\gamma).$$

13.21 Equation of the tangent at the point whose vectorial angle is α.

If the points on the conic

$$\frac{l}{r}=1+e\cos\theta,$$

whose vectorial angles are $\alpha-\beta$, $\alpha+\beta$ coincide, β becomes zero, and in this limiting position the chord becomes a tangent to the conic at the point whose vectorial angle is α.

The equation of the tangent to the conic at the point whose vectorial angle is α is, therefore,

$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta$$

For the conic $\frac{l}{r}=1+e\cos(\theta-\gamma)$, the equation of the tangent at the point 'α' is

$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos(\theta-\gamma).$$

⊥ to tangent

13.22 Normal. The equation of the tangent at the point (r', α) on the conic $\frac{l}{r}=1+e\cos\theta$ is

$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta. \qquad ...(1)$$

The equation of the normal which is perpendicular to the tangent is of the form

$$\frac{A}{r}=\cos\left(\theta+\frac{\pi}{2}-\alpha\right)+e\cos\left(\theta+\frac{\pi}{2}\right),$$

i.e., $$\frac{A}{r}=-\sin(\theta-\alpha)-e\sin\theta. \qquad ...(2)$$

Since (2) passes through (r', α),

$$\frac{A}{r'}=-e\sin\alpha. \qquad ...(3)$$

Now from the equation of the conic,

$$\frac{l}{r'}=1+e\cos\alpha.$$

Hence, from (3) $A=\dfrac{-el\sin\alpha}{1+e\cos\alpha}$.

Substituting in (2), the equation of the normal at the point whose vectorial angle is α, is

$$\frac{el\sin\alpha}{(1+e\cos\alpha)r}=\sin(\theta-\alpha)+e\sin\theta.$$

Examples

1. Find the condition that the line $l/r=A\cos\theta+B\sin\theta$ may be a tangent to the conic $l/r=1+e\cos\theta$.

(*Agra*, 1973; *Punjab*, 1977)

Ans. $(A-e)^2+B^2=1$.

Hint. Compare the equation of the line with the equation of gent at the point 'α'.

2. Chords of a conic subtend a constant angle 2α at the focus. Find the locus of the point where the chord are met by the internal bisector of the angle which they subtend at the focus.

(*Agra*, 1967; *Rajasthan*, 1967)

Solution. Let the equation of the conic be $l/r=1+e\cos\theta$ and the vectorial angles of the extremities of the chord be $\beta-\alpha$ and

$\beta+\alpha$. This chord then subtends an angle 2α at the focus and its equation is

$$\frac{l}{r}=\sec\alpha\cos(\theta-\beta)+e\cos\theta. \qquad ...(1)$$

If (r', θ') be the coordinates of the point where the chord is meet by the internal bisector of the angle which it subtends at the focus, then

$$\theta'=\beta. \qquad ...(2)$$

Since (r', θ') lies on (1),

$$\frac{l}{r'}=\sec\alpha\cos(\theta'-\beta)+e\cos\theta'. \qquad ...(3)$$

From (2) and (3), we obtain

$$\frac{l}{r'}=\sec\alpha+e\cos\theta'$$

or $$\frac{l\cos\alpha}{r'}=1+e\cos\alpha\cos\theta'.$$

Hence the locus of (r', θ') is the conic

$$\frac{l\cos\alpha}{r}=1+e\cos\alpha\cos\theta.$$

3. If S be the focus and P, Q be two points on the conic such that the angle PSQ is constant and equal to 2δ, prove that the locus of the point of intersection of tangents at P and Q is a conic whose focus is S.

(*Agra*, 1962)

4. Prove that the equation to the locus of the foot of the perpendicular from the focus of a conic $l/r=1+e\cos\theta$ on any tangent to it is

$$r^2(e^2-1)-2ler\cos\theta+l^2=0.$$

Discuss the particular case when $e=1$.

(*Rajasthan*, 1971; *Agra*, 1973; *Kanpur*, 1975)

5. If the normal at L, an extremity of the latus rectum of the conic $l/r=1+e\cos\theta$ meet the conic again at Q, show that

$$SQ=\frac{l(1+3e^2+e^4)}{1+e^2-e^4}.$$

Solution. The polar coordinates of the point L are $(l, \frac{1}{2}\pi)$, and e equation of the normal at L is

$$\frac{l}{r}\cdot\left(\frac{e\sin\frac{1}{2}\pi}{1+e\cos\frac{1}{2}\pi}\right)=e\sin\theta+\sin(\theta-\tfrac{1}{2}\pi)$$

or $$\frac{le}{r}=e\sin\theta-\cos\theta. \qquad ...(1)$$

Eliminating θ between (1) and the equation of the conic,

$$\left\{\frac{el}{r}+\frac{l-r}{er}\right\}^2=e^2\left\{1-\left(\frac{l-r}{er}\right)^2\right\}$$

which gives on simplification

$$(l-r)\,[(l-r)(1+e^2)+2le^2+e^4(l+r)]=0.$$

The value $r=l$ corresponds to the point L. From the other factor, we obtain

$$SQ=r=\frac{l(1+3e^2+e^4)}{1+e^2-e^4}.$$

This proves the proposition.

6. The normal to the conic $l/r=1+e\cos\theta$ at the point whose vectorial angle is 'α' meets the curve again at the point whose vectorial angle is 'β'. Prove that

$$\tan\tfrac{1}{2}\beta=-\frac{1+2e\cos^2\frac{1}{2}\alpha+e^2}{1-2e\sin^2\frac{1}{2}\alpha+e^2}\cot\tfrac{1}{2}\alpha.$$

7. If the normals at the points whose vectorial angle are α, β, γ on the parabola $l/r=1+\cos\theta$ meet in a point (ρ, ϕ), show that $2\phi=\alpha+\beta+\gamma$.

(*Allahabad*, 1968; *Agra*, 1972)

Solution. The equation of the normal at a point on the parabola whose vectorial angle is θ_1, is

$$\frac{l}{r}\cdot\left(\frac{\sin\theta_1}{1+\cos\theta_1}\right)=\sin\theta+\sin(\theta-\theta_1).$$

If this passes through the point (ρ, ϕ), then

$$\frac{l\sin\theta_1}{\rho(1+\cos\theta_1)}=\sin\phi+\sin(\phi-\theta_1)$$

or $$\frac{2l\sin\frac{1}{2}\theta_1\cos\frac{1}{2}\theta_1}{2\rho.\cos^2\frac{1}{2}\theta_1}=\sin\phi(\phi+\cos\theta_1)-\cos\phi\sin\theta_1$$

$$=2\sin\phi\cos^2\tfrac{1}{2}\theta_1-2\cos\phi\sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_1$$

or $$\frac{l}{\rho}\tan^2\tfrac{1}{2}\theta_1+\left[\frac{l}{\rho}+2\cos\phi\right]\tan\tfrac{1}{2}\theta_1-2\sin\phi=0. \qquad ...(2)$$

This is a cubic equation in $\tan\frac{1}{2}\theta_1$. If $\tan\frac{1}{2}\alpha$, $\tan\frac{1}{2}\beta$, $\tan\frac{1}{2}\gamma$ be the three roots of this equation, we have

$$\tan(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2}\gamma)=\frac{(-2\rho\sin\phi)/l}{1-[(l+2\rho\cos\phi)/l]}=\tan\phi.$$

The result now follows.

13.3 Polar. Let (r_1, θ_1) be a given point and let us find the equation to its polar with respect to the conic $\frac{l}{r}=1+e\cos\theta$.

We shall use the property that the polar of a point is the chord of contact of tangents drawn from it to the conic.

If $\alpha-\beta$, $\alpha+\beta$ be the vectorial angles of the points of contact, the equation of the chord of contact is

$$\frac{l}{r}=\sec\beta\cos(\theta-\alpha)+e\cos\theta. \qquad \ldots(1)$$

Now the equation of the tangent at '$\alpha-\beta$' is

$$\frac{l}{r}=\cos(\theta-\alpha+\beta)+e\cos\theta.$$

This passes through (r_1, θ_1). Therefore,

$$\frac{l}{r_1}=\cos(\theta_1-\alpha+\beta)+e\cos\theta_1. \qquad \ldots(2)$$

Similarly, $$\frac{l}{r_1}=\cos(\theta_1-\alpha-\beta)+e\cos\theta_1. \qquad \ldots(3)$$

From (2) and (3),

$$\cos(\theta_1-\alpha+\beta)=\cos(\theta_1-\alpha-\beta),$$

i.e., $$\theta_1-\alpha+\beta=\pm(\theta_1-\alpha-\beta).$$

Since $\beta\neq 0$, $$\theta_1-\alpha+\beta=-(\theta_1-\alpha-\beta),$$

or $$\alpha=\theta_1.$$

Substituting this value of α in (2) or (3), we get

$$\cos\beta=\frac{l}{r_1}-e\cos\theta_1.$$

From (1), therefore, the polar of (r_1, θ_1) is

$$\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)=\cos(\theta-\theta_1).$$

13.4 Miscellaneous solved examples. We shall give below a few solved examples.

1. *Show that the* director ~~*diameter*~~ *circle of the conic* $\frac{l}{r}=1+e\cos\theta$ *is*

$$r^2(1-e^2)+2elr\cos\theta-2l^2=0.$$

(*Agra*, 1973; *Kanpur*, 1975; *Lucknow*, 1965, 1978)

Solution. The equations of the tangents at the points 'α', 'β' of the given conic, are

$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta,$$

and $$\frac{l}{r}=\cos(\theta-\beta)+e\cos\theta.$$

If θ be the vectorial angle of the point where the tangents intersect each other,

$$\cos(\theta-\alpha)=\cos(\theta-\beta),$$

i.e., $$\theta-\alpha=\pm(\theta-\beta).$$

Neglecting the plus sign,

$$\theta=\frac{\alpha+\beta}{2} \qquad \ldots(1)$$

Substituting this value of θ in the equation of either tangent the radius vector r of the point of intersection can be written as

$$\frac{l}{r}=\cos\frac{\alpha-\beta}{2}+e\cos\frac{\alpha+\beta}{2}. \qquad \ldots(2)$$

Converting the equations of the tangents in Cartesian coordinates we see that they are at right angles if

$$(\cos\alpha+e)(\cos\beta+e)+\sin\alpha\sin\beta=0,$$

i.e., if $$e^2+e(\cos\alpha+\cos\beta)+\cos(\alpha-\beta)=0,$$

which can be written as

$$e^2+2e\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\cos^2\frac{\alpha-\beta}{2}-1=0. \qquad \ldots(3)$$

Eliminating α and β from (3) with the help of (1) and (2),

$$e^2+2e\cos\theta.\left(\frac{l}{r}-e\cos\theta\right)+2\left(\frac{l}{r}-e\cos\theta\right)-1=0,$$

or $$r^2(1-e^2)+2elr\cos\theta-2l^2=0,$$

which is the required equation of the director circle.

2. *Prove that the two conics*

$$\frac{l_1}{r}=1-e_1\cos\theta,$$

and $$\frac{l_2}{r}=1-e_2\cos(\theta-\alpha)$$

will touch one another if

$$l_1^2(1-e_2^2)+l_2^2(1-e_1^2)=2l_1l_2(1-e_1e_2\cos\alpha).$$

(*Rajasthan*, 1970; *Agra*, 1977; *Lucknow*, 1960)

Solution. If the vectorial angle of the point of contact be θ', the equations of the tangents of the given conics are respectively

$$\frac{l_1}{r}=\cos(\theta-\theta')-e_1\cos\theta,$$

and $$\frac{l_2}{r}=\cos(\theta-\theta')-e_2\cos(\theta-\alpha),$$

i.e., $$\frac{l_1}{r}=\cos(\theta'-e_1)\cos\theta+\sin\theta\sin\theta',$$

and $$\frac{l_2}{r}=(\cos\theta'-e_2\cos\alpha)\cos\theta+(\sin\theta'-e_2\sin\alpha)\sin\theta.$$

Comparing coefficients,

$$\frac{l_2}{l_1}=\frac{\cos\theta'-e_2\cos\alpha}{\cos\theta'-e_1}=\frac{\sin\theta'-e_2\sin\alpha}{\sin\theta'}.$$

From these we get

$$\sin\theta'(l_2-l_1)=-e_2l_1\sin\alpha,$$

and $$\cos\theta'(l_2-l_1)=-e_1l_2-e_2l_1\cos\alpha.$$

Squaring and adding we get the desired result.

3. *A conic is described having the same focus and eccentricity as the conic* $\frac{l}{r}=1+e\cos\theta$, *and the two conics touch at the point* $\theta=\alpha$; *prove that the length of its latus rectum is*

$$\frac{2l(1-e^2)}{e^2+2e\cos\alpha+1}.$$ (*Kanpur*, 1974; *I. A. S.*, 1964)

Solution. Let the other conic be

$$\frac{l'}{r}=1+e\cos(\theta-\gamma).$$

The tangents to the two conics at $\theta=\alpha$ are respectively

$$\frac{l}{r}=\cos(\theta-\alpha)=e\cos\theta$$

and $$\frac{l'}{r}=\cos(\theta-\alpha)+e\cos(\theta-\gamma).$$

Comparing coefficients,

$$\frac{l'}{r}=\frac{\cos\alpha+e\cos\gamma}{\cos\alpha+e}=\frac{\sin\alpha+e\sin\gamma}{\sin\alpha}.$$

From these

$$e\cos\gamma=\left(\frac{l'}{r}-1\right)\cos\alpha+\frac{el'}{l}$$

and $$e\sin\gamma=\left(\frac{l'}{l^2}-1\right)\sin\alpha.$$

Squaring and adding,

$$\frac{l'^2}{l^2}(1+2e\cos\alpha+e^2)-\frac{2l'}{l}(1+e\cos\alpha)+1-e^2=0.$$

Solving $$\frac{l'}{l}=1 \text{ or } \frac{1-e^2}{1+2e\cos\alpha+e^2}.$$

But $\frac{l'}{l}\neq 1$. This gives the result.

4. *P, Q, R are three points on the conic*

$$\frac{l}{r}=1+e\cos\theta,$$

the focus S being the pole; SP and SR meet the tangent at Q in M and N so that SM=SN=l. Prove that PR touches the conic

$$\frac{l}{r}=1+2e\cos\theta.$$ (*Agra*, 1975; Rohilkhand, 1977)

Solution. Let the vectorial angles of P, Q, R be α, β, γ. The equation to PR then is

$$\frac{l}{r}=\sec\frac{\alpha-\beta}{2}\cos\left(\theta-\frac{\alpha+\beta}{2}\right)+e\cos\theta. \quad ...(1)$$

The equation to the tangent at Q is

$$\frac{l}{r}=\cos(\theta-\gamma)+e\cos\theta.$$

Two points on this are (l,α) and (l,β). Therefore,

$$1=\cos(\alpha-\gamma)+e\cos\alpha, \quad ...(2)$$

and $$1=\cos(\beta-\gamma)+e\cos\beta. \quad ...(3)$$

(2) and (3) can be written as

$$(\cos\gamma+e)\cos\alpha+\sin\alpha\sin\gamma=1 \quad ...(4)$$

and $$(\cos\gamma+e)\cos\beta+\sin\beta\sin\gamma=1. \quad ...(5)$$

Eliminating $\sin\gamma$ between (4) and (5),

$$(\cos\gamma+e)\sin(\alpha-\beta)=\sin\alpha-\sin\beta,$$

i.e., $$\cos\gamma+e=\sec\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}. \quad ...(6)$$

Subtracting (5) from (4),

$$\sin\gamma(\sin\alpha-\sin\beta)+(\cos\gamma+e)(\cos\alpha-\cos\beta)=0,$$

i.e., $$\sin\gamma=\sec\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}. \quad ...(7)$$

Writing equation (1) as

$$\frac{l}{r}=\sec\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}\cos\theta$$
$$+\sec\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}\sin\theta+e\cos\theta,$$

and substituting from (6) and (7), the equation to PR becomes

$$\frac{l}{r}=\cos(\theta-\gamma)+2e\cos\theta,$$

which is obviously a tangent to the conic $\frac{l}{r}=1+2e\cos\theta$, at the point '$\gamma$'.

5. *An ellipse and a parabola have a common focus S and intersect in two real points P and Q of which P is the vertex of the parabola. If e be the eccentricity of the ellipse and α the angle which SP makes with the major axis, prove that*

$$\frac{SQ}{SP}=1+\frac{4e^2\sin^2\alpha}{(1-e\cos\alpha)^2}.$$

Solution. Let the equations of parabola and ellipse be

$$\frac{l}{r}=1+\cos\theta \text{ and } \frac{L}{r}=1+e\cos(\theta-\alpha).$$

$\therefore$
$$SP=\frac{l}{2}=\frac{L}{1+e\cos\alpha}. \qquad ...(1)$$

Let β the vectorial angle of Q, then

$$\frac{l}{SQ}=1+\cos\beta \text{ and } \frac{L}{SQ}=1+e\cos(\beta-\alpha).$$

Therefore,
$$\frac{1+\cos\beta}{l}=\frac{1+e\cos(\beta-\alpha)}{L},$$

or, from (1),

$$\frac{1+\cos\beta}{2}=\frac{1+e\cos(\beta-\alpha)}{(1+e\cos\alpha)},$$

i.e.
$$\cos^2\frac{\beta}{2}(1+e\cos\alpha)=1+e\left[\cos\alpha\left(2\cos^2\frac{\beta}{2}-1\right)\right.$$
$$\left.+\sin\alpha.\,2\sin\frac{\beta}{2}\cos\frac{\beta}{2}\right]$$

or
$$\sin^2\frac{\beta}{2}+e\cos\alpha\cos^2\frac{\beta}{2}-e\cos\alpha+2e\sin\alpha.\sin\frac{\beta}{2}\cos\frac{\beta}{2}=0,$$

$$\tan^2\frac{\beta}{2}+e\cos\alpha-e\cos\alpha\left(1+\tan^2\frac{\beta}{2}\right)$$
$$+2e\sin\alpha\tan\frac{\beta}{2}=0,$$

or
$$\tan^2\frac{\beta}{2}(1-e\cos\alpha)+2e\sin\alpha\tan\frac{\beta}{2}=0,$$

Rejecting $\beta=0$,

$$\tan\frac{\beta}{2}=-\frac{2e\sin\alpha}{1-e\cos\alpha}.$$

Now,
$$\frac{SQ}{l}=\frac{1}{1+\cos\beta}$$

i.e., from (1),
$$\frac{SQ}{SP}=1+\tan^2\frac{\beta}{2}=1+\frac{4e^2\sin^2\alpha}{(1-e\cos\alpha)^2}.$$

6. *If A, B, C be any three points on a parabola and the tangent at these points from a triangle $A'B'C'$, show that*

$$SA.SB.SC=SA'.SB'.SC',$$

S being the focus of the conic. (*Agra*, 1966)

Solution. Let the equation of the parabola be

$$\frac{l}{r}=1+\cos\theta.$$

If α, β, γ be the vectorial angles of the point A, B, C, then the radii vectors of the points of intersection of tangents a A', B', C' are respectively

$$SA'=\tfrac{1}{2}l\sec\tfrac{1}{2}\beta\sec\tfrac{1}{2}\gamma,\ SB'=\tfrac{1}{2}l\sec\tfrac{1}{2}\gamma\sec\tfrac{1}{2}\alpha$$

and
$$SC'=\tfrac{1}{2}l\sec\tfrac{1}{2}\alpha\sec\tfrac{1}{2}\beta.$$

Now $SA=\frac{1}{2}l\sec^2\frac{1}{2}\alpha$, $SB=\frac{1}{2}l\sec^2\frac{1}{2}\beta$ and $SC=\frac{1}{2}l\sec^2\frac{1}{2}\gamma$.

Consequently
$$SA'.SB'.SC'=\tfrac{1}{8}l^3\sec^2\tfrac{1}{2}\alpha\sec^2\tfrac{1}{2}\beta\sec^2\tfrac{1}{2}\gamma$$
$$=SA.SB\,SC,$$

which proves the proposition.

7. *Show that the polar equation of the circle circumscribing the triangle formed by the tangents to the parabola $l/r=1+\cos\theta$ at the points where vectorial angles are 2α, 2β and 2γ, is*

$$2r\cos\alpha\cos\beta\cos\gamma=l\cos(\theta-\alpha-\beta-\gamma).$$ (*Allahabad*, 1972)

Solution. The coordinates of the points of intersection of tangents at the points whose vectorial angles are 2α, 2β, 2γ, are

$$P:\{\tfrac{1}{2}l\sec\beta\sec\gamma,\ (\beta+\gamma)\},\ Q:\{\tfrac{1}{2}l\sec\gamma\sec\alpha,\ (\gamma+\alpha)\}$$

and $R:\{\frac{1}{2}l\sec\alpha\sec\beta,\ (\alpha+\beta)\}$ respectively.

Let the equation of the circumcircle of the triangle PQR be

$$r=d\cos(\theta-\lambda) \qquad ...(1)$$

where d is the diameter of the circle. If P, Q, R lie on (1), then

$$\tfrac{1}{2}l\sec\beta\sec\gamma=d\cos(\beta+\gamma-\lambda), \qquad ...(2)$$

$$\tfrac{1}{2}l \sec\gamma \sec\alpha = d\cos(\gamma+\alpha-\lambda), \qquad ...(3)$$

and
$$\tfrac{1}{2}l \sec\alpha \sec\beta = d\cos(\alpha+\beta-\lambda). \qquad ...(4)$$

Dividing (2) and (3), we obtain

$$\frac{\sec\beta}{\sec\alpha}=\frac{\cos\alpha}{\cos\beta}=\frac{\cos(\beta+\gamma-\lambda)}{\cos(\gamma+\alpha-\lambda)}$$

which gives $\lambda=\alpha+\beta+\gamma$.

Consequently from (2), we get

$$d=\tfrac{1}{2}l\sec\alpha\sec\beta\sec\gamma.$$

Putting the values of d and λ in (1), the equation of the circum-circle of the triangle is

$$2r\cos\alpha\cos\beta\cos\gamma=l\cos(\theta-\alpha-\beta-\gamma).$$

Examples on Chapter XIII

1. In any conic prove that

(i) the tangents at the ends of a focal chord meet on the directrix. (*Punjab*, 1974)

(ii) the portion of tangent intercepted between the curve and the directrix subtends a right angle at the corresponding focus. (*Allahabad*, 1970)

2. PSP' is a focal chord of a conic. Prove that the angle between the tangents at P and P', is

$$\tan^{-1}\left(\frac{2e\sin\alpha}{1-e^2}\right)$$

where α is the angle between chord and the initial line.

(*Allahabad*, 1973; *U. P. C. S.*, 1973)

3. The tangents at P and Q to a parabola meet at T. Show that

$$ST^2=SP.SQ$$

where S is the focus of the parabola. (*Lucknow*, 1978)

4. Find the equation of the chord of the conic $l/r=1+e\cos\theta$, joining the points whose vectorial angles are 30° and 90°.

(*Gorakhpur*, 1967)

Ans. $l/r=\{e+1/\sqrt{3}\}\cos\theta+\sin\theta$.

5. Tangents are drawn at the extremities of perpendicular focal radii of a conic. Show that the locus of their point of intersection is another conic having the same focus.

6. Prove that, if chords of a conic subtend a constant angle at a focus, the tangents at the ends of the chords meet on a fixed conic and these chords will touch another fixed conic.

(*Lucknow*, 1969; *Rohilkhand*, 1976)

7. Find the locus of the pole of a chord which subtends a constant angle 2α at a focus of a conic, distinguishing between the cases for which $\cos\alpha >, =, < e$.

(*Kanpur*, 1974; *Gorakhpur*, 1974; *Lucknow*, 1980)

Ans. $\dfrac{l\sec\alpha}{r}=1+e\sec\alpha\cos\theta$.

8. Show that the equation to the circle, which passes through the focus and touches the conic $l/r=1-e\cos\theta$ at the point $\theta=\alpha$, is $r(1-e\cos\alpha)^2=l[\cos(\theta-\alpha)-e\cos(\theta-2\alpha)]$. (*Rajasthan*, 1966)

9. If the tangents from P subtend a fixed angle β at the focus S of a conic, prove that the locus of the middle point of SP is a conic of eccentricity $e\sec\beta$, and find its latus rectum.

10. Two equal ellipse of eccentricity e are placed with their axes at right angles and have a common focus S. If PQ be common tangent to two ellipse, show that the angle PSQ is

$$2\sin^{-1}\left(\frac{e}{\sqrt{2}}\right).$$ (*Kanpur*, 1973; *Lucknow*, 1978)

Hint. Take the equations of the ellipse as

$$\frac{l}{r}=1+e\cos\theta, \qquad \frac{l}{r}=1+e\cos(\theta-\tfrac{1}{2}\pi).$$

11. Two chords QP, PR of a conic subtend equal angles at the focus. Prove that the chord QR and the tangent at P intersect on the directrix.

12. If the normals at three points of a parabola $r=a\operatorname{cosec}^2\tfrac{1}{2}\theta$, whose vectorial angles are α, β, γ meet in a point whose vectorial angle is ϕ, prove that $2\phi=\alpha+\beta+\gamma-\pi$.

(*Agra*, 1962; *Gorakhpur*, 1967)

13. Show that the equation to the circle circumscribing the triangle formed by the three tangents to the parabola $2a/r=1-\cos\theta$, drawn at the points whose vectorial angles are α, β, γ, is

$$r=a\operatorname{cosec}\frac{\alpha}{2}\operatorname{cosec}\frac{\beta}{2}\operatorname{cosec}\frac{\gamma}{2}\sin\left(\frac{\alpha+\beta+\gamma}{2}-\theta\right),$$

and hence it always passes through the focus of the parabola.

14. If the normals at the points whose vectorial angles are $\alpha, \beta, \gamma, \delta$ on the conic $l/r=1+e\cos\theta$ are concurrent, prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}+\left(\frac{1+e}{1-e}\right)^2=0.$$

Also prove that if the four normals meet in a point (ρ, ϕ),

$$\alpha+\beta+\gamma+\delta-2\phi=\text{odd multiple of }\pi.$$ (*U. P. C. S.*, 1972)

Hint. The equation of the normal at 'λ' is

$$\frac{l}{r}\cdot\frac{e\sin\lambda}{1+e\cos\lambda}=\sin(\theta-\lambda)+e\sin\theta.$$

Express in terms of $\tan\dfrac{\lambda}{2}$.

15. If $\theta_1, \theta_2, \theta_3, \theta_4$ be the vectorial angles of the points where a circle through the focus cuts the conic $l/r=1+e\cos\theta$, prove that

$$\sin\theta_1+\sin\theta_2+\sin\theta_3+\sin\theta_4=0. \qquad (I.\ A.\ S.,\ 1968)$$

16. If the tangents at any two points P and Q of a conic meet in a point T and if the straight line PQ meets the directrix corresponding to focus S in a point K, prove that $\angle KST$ is a right angle.

(*Allahabad*, 1966; *Jodhpur*, 1968)

17. If PQ is the chord of contact of tangents drawn from a point T to an ellipse $l/r=1+e\cos\theta$, whose focus is S, prove that

$$\frac{1}{SP.SQ}-\frac{1}{ST^2}=\frac{1}{b^2}\sin^2\tfrac{1}{2}PSQ,$$

where $2b$ is the minor axis of the ellipse.

18. *Show that the equation of pair of tangents to a conic from a pair* (r_1, θ_1) *is* $(S^2-1)(S_1^2-1)=P^2$,

where $S\equiv\dfrac{l}{r}-e\cos\theta;\ S_1\equiv\dfrac{l}{r_1}-e\cos\theta_1$

and $P\equiv\left\{\left(\dfrac{l}{r}-e\cos\theta\right)\left(\dfrac{l}{r_1}-e\cos\theta_1\right)-\cos(\theta-\theta_1)\right\}.$

Hence prove that the asymptotes of the conic are

$$\frac{el}{r}=(e^2-1)\cos\theta\pm\sqrt{(e^2-1)}\sin\theta. \qquad (Gorakhpur,\ 1975)$$

Solution. The equation of the conic is

$$\frac{l}{r}=1+e\cos\theta. \qquad \ldots(1)$$

The equation of any tangent to (1) at the point 'α' is

$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta. \qquad \ldots(2)$$

It this tangent passes through (r_1, θ_1), then

$$\frac{l}{r_1}=\cos(\theta_1-\alpha)+e\cos\theta_1 \qquad \ldots(3)$$

The equation of pair of tangent is obtained by eliminating α from (2) and (3). From (2) we get

$$S^2-1=\left(\frac{l}{r}-e\cos\theta\right)^2-1=\cos^2(\theta-\alpha)-1$$
$$=-\sin^2(\theta-\alpha), \qquad \ldots(4)$$

and similarly, from (3) we get

$$S_1^2-1=\left(\frac{l}{r_1}-e\cos\theta_1\right)^2-1=\cos^2(\theta_1-\alpha)-1$$
$$=-\sin^2(\theta_1-\alpha). \qquad \ldots(5)$$

Again. from (2) and (3), we obtain

$$\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)=\cos(\theta-\alpha)\cos(\theta_1-\alpha)$$

or
$$P\equiv\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)-\cos(\theta-\theta_1)$$
$$=\cos(\theta-\alpha)\cos(\theta_1-\alpha)-\cos(\theta-\theta_1)$$
$$=\tfrac{1}{2}[\cos(\theta+\theta_1-2\alpha)+\cos(\theta-\theta_1)]-\cos(\theta-\theta_1)$$
$$=\tfrac{1}{2}[\cos(\theta+\theta_1-2\alpha)-\cos(\theta-\theta_1)]$$
$$=-\sin(\theta-\alpha)\sin(\theta_1-\alpha). \qquad \ldots(6)$$

From (4), (5) and (6), we get the equation of pair of tangents as

$$(S^2-1)(S_1^2-1)=P^2. \qquad \ldots(7)$$

Again, the asymptotes are pair of tangents through the centre whose polar coordinates referred to focus as pole are $(ae, 0)$ and the axis of the conic as the initial line. Consequently, writing $r_1=ae$, $\theta_1=0$, we have

$$S_1=\frac{l}{r_1}-e\cos\theta_1=\frac{l}{ae}-e$$
$$=\frac{a^2(e^2-1)}{a^2e}-e, \text{ where } l=b^2/a \text{ and } b^2=a^2(e^2-1).$$
$$=-\frac{1}{e}.$$

and
$$P\equiv\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)-\cos(\theta-\theta)$$
$$=\left(\frac{l}{r}-e\cos\theta\right)\left(-\frac{1}{e}\right)\cos\theta.$$

Putting the values of S, S_1 and P in (7), we obtain

$$\left[\left(\frac{l}{r}-e\cos\theta\right)^2-1\right]\left(\frac{1}{e^2}-1\right)$$

$$=\left[\left(\frac{l}{r}-e\cos\theta\right)\left(-\frac{1}{e}\right)-\cos\theta\right]$$

or $$\frac{l^2e^2}{r^2}-2(e^2-1)\cos\theta\,\frac{el}{r}+(e^2-1)^2\cos^2\theta=(e^2-1)\,(1-\cos^2\theta)$$

or $$\left[\frac{el}{r}-(e^2-1)\cos\theta\right]^2=(e^2-1)\sin^2\theta$$

or $$\frac{el}{r}-(e^2-1)\cos\theta=\pm\sqrt{(e^2-1)}\sin\theta$$

or $$\frac{el}{r}=(e^2-1)\cos\theta\pm\sqrt{(e^2-1)}\sin\theta.$$

19. A is the vertex of a conic; and AP a chord which meets the latus rectum in Q. A parallel chord $P'SQ'$ is drawn through the focus S. Show that the ratio $\dfrac{AP.AQ}{SQ'.SP'}$ is constant.

20. PQ is a chord of an ellipse, one of whose foci is S, and PQ passes through a fixed point O. Show that the product

$\tan(\frac{1}{2}PSO)\tan(\frac{1}{2}QSO)$ is constant.

21. If the circle $r+2a\cos\theta=0$ intersects the conic $l/r=1+e\cos(\theta-\alpha)$ in four points and the algebraic sum of the focal distances of these four points is equal to $2a$, show that the eccentricity of the conic is equal to $\cos\alpha$.

22. A straight line drawn through the common focus S of a number of conics meets them in point P_1, P_2, P_3,......On it is taken a point Q such that reciprocal of SQ is equal to the sum of the reciprocals of SP_1, SP_2, SP_3,......Prove that the locus of Q is a conic section whose focus is S and the reciprocal of whose latus rectum is equal to the sum of the reciprocals of the latera-recta of the given conics.

23. *Two conics have a common focus. Prove that two of their common chords will pass through the point of intersection of their directrices.*

Solution. Let the equation of the two conics be

$$\frac{l}{r}=1+e_1\cos\theta \qquad ...(1)$$

and $$\frac{k}{r}=1+e_2\cos(\theta-\alpha). \qquad ...(2)$$

Equation (1) can also be written as (Ex. 1 of § 13.13)

$$\frac{l}{r}=-1+e_1\cos\theta. \qquad ...(3)$$

Adding (2) and (3), and subtracting (1) and (2), we obtain

$$\frac{l\pm k}{r}=\cos\theta\,(e_1\pm e_2\cos\alpha)\pm e_2\sin\theta\sin\alpha \qquad ...(4)$$

which represents two straight lines.

The equation (4) can also be written as

$$\frac{l}{r}-e_1\cos\theta\pm\left[\frac{k}{r}-e_2\{\cos(\theta-\alpha)\}\right]=0$$

which shows that two common chords of conics given by (4) pass through the intersection of

$$\frac{l}{r}=e_1\cos\theta \text{ and } \frac{k}{r}=e_2\cos(\theta-\alpha)$$

which are the directrices of the given conics.

CHAPTER XIV

CONICS IN GENERAL, DOUBLE CONTACT

14.1 Intersection of two conics. Let

$$ax^2+2hxy+by^2+2gx+2fy+c=0, \qquad ...(1)$$

and
$$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0, \qquad ...(2)$$

be the equations of two given conics.

Writing equations (1) and (2) in descending powers of x, we get

$$ax^2+2x\,(hy+g)+by^2+2fy+c=0, \qquad ...(3)$$

and
$$a'x^2+2x\,(h'y+g')+b'y^2+2f'y+c'=0. \qquad ...(4)$$

Elimination of x between (3) and (4) will give a fourth degree equation in y. The equation in y will have four roots.

Also, from (3) and (4),

$2x\,\{a'\,(hy+g)-a\,(h'y+g')\}+a'\,(by^2+2fy+c)-a\,(b'y^2+2f'y+c')=0$ from which we see that corresponding to each value of y there exists only one value of x.

Two conics, therefore, intersect in four points, which may be real or imaginary.

Further, since we can have at most two pairs of coincident points of intersection, two conics cannot touch each other at more than two distinct points.

14.2 Conic through five points. We know that the general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a conic. Now this equation seems to contain six constants a, b, c, f, g, h, but the actual number of constants is only five as can easily be seen on dividing by one of the constants.

To evaluate the five constants uniquely we need five independent equations containing them. If the conic passes through five given points, we shall get five equations which may or may not be independent. For example, if four of the given points lie on one straight line, the equations will not be independent. To show this we take the line joining the four collinear points as one coordinate axis, say x-axis. The coordinates of these points can then be written as $(0, 0)$, $(\alpha, 0)$, $(\alpha', 0)$ and $(\alpha'', 0)$.

Since the conic passes through $(0, 0)$, $c=0$. Similarly, since it passes through each of the three other points, we have

$$a\alpha^2+2g\alpha=0, \qquad ...(1)$$

$$a\alpha'^2+2g\alpha'=0, \qquad ...(2)$$

and
$$a\alpha''^2+2g\alpha''=0. \qquad ...(3)$$

From (1) and (2), $a=0$, $g=0$. Equation (3) is now also satisfied, and the equation to the conic reduces to

$$2hxy+by^2+2fy=0.$$

If the coordinates of the fifth point through which the conic passes be (β, γ), we have

$$2h\beta\gamma+b\gamma^2+2f\gamma=0$$

as the equation to determine the two ratios $\dfrac{h}{b}$, $\dfrac{f}{b}$.

One of the two ratios $h : b : f$ can therefore be given any arbitrary value and we conclude that *an infinite number of conics can be drawn through five points, four of which are collinear.* Each conic in this case consists of a pair of straight lines, one of which passes through the four given points and the other through the fifth point.

If only three of the given points are collinear, the conic will evidently be a pair of straight lines of which one passes through three collinear points and the other through the remaining two points.

We thus have only one conic passing through five given points, no four of which are collinear.

14.3 Conic through the intersection of two given conics.

Let
$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

and
$$S'\equiv a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$$

be two given conics.

Consider the equation

$$S+\lambda S'=0, \qquad ...(1)$$

which being of the second degree represents a conic. It is satisfied by the coordinates of the four points of intersection of $S=0$, $S'=0$ and consequently is the equation of a conic through four given points.

If the fifth point through which (1) passes is known, the constant λ is determined uniquely and equation (1) then represents a fixed conic.

14.4 Some solved examples.

1. *Prove that all conics through the intersections of two rectangular hyperbolas are themselves rectangular hyperbolas.* (Gorakhpur, 1968)

Let the conics $S=0$, $S'=0$ be rectangular hyperbolas. Then, $a+b=0$, $a'+b'=0$. In the conic $S+\lambda S'=0$, the sum of the coefficients of x^2 and y^2 is $a+b+\lambda(a'+b')$, which is identically equal to zero in virtue of the above relations.

Hence the conic $S+\lambda S'=0$ which passes through the intersection of $S=0, S'=0$ is a rectangular hyperbola.

2. *The normal to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at the ends of the chords* $lx+my-1=0$ *and* $l'x+m'y-1=0$ *are concurrent. Show that*

$$a^2ll'=b^2mm'=-1.$$

The equation to the conic passing through the four feet of the normals is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(lx+my-1)(l'x+my-1)=0. \quad ...(1)$$

Now, the feet of normals from a given point (h, k) to the ellipse lie on the rectangular hyperbola (see Chapter X)

$$xy\left(\frac{1}{b^2}-\frac{1}{a^2}\right)-\frac{h}{b^2}y+\frac{k}{a^2}x=0.$$

Therefore in equation (1) the cosntant term and the coefficients of x^2 and y^2 must separately vanish.

This gives $\lambda-1=0$

$$\lambda ll'-\frac{1}{a^2}=0$$

and $\lambda mm'+\frac{1}{b^2}=0.$

Hence, $a^2ll'=b^2mm'=-1.$

3. *Prove that the chord af contact of tangents from a fixed point to a system of conics through four given points passes through a fixed point.* (*Lucknow*, 1978)

Let the fixed point be taken as origin, and let $S=0$, $S'=0$ be two conics which intersect in four given points. Then the equation of any conic of the system is

$$S+\lambda S'=0. \quad ...(1)$$

The chord of contact of tangents from (0, 0) to (1) is

$$gx+fy+c+\lambda(g'x+f'y+c')=0.$$

This passes through the point of intersection of $gx+fy+c=0$ and $g'x+f'y+c'=0$, which is a fixed point.

4. *Prove that the common chords of a conic and circle taken in pairs are equally inclined to the axes of the conic.* (*Lucknow*, 1978)

Let us take the coordinate axes parallel to the axis of the conic. The equation of the conic then is

$$ax^2+by^2+2gx+2fy+c=0, \quad ...(1)$$

a or b being *zero* if the conic is a parabola. Let the common chords of (1) and a circle be

$$lx+my+n=0, \quad ...(2)$$

$$l'x+m'y+n'=0. \quad ...(3)$$

Now

$$ax^2+by^2+2gx+2fy+c+\lambda(lx+my+n)(l'x+m'y+n')=0 \quad ...(4)$$

is the equation of a conic through the four points of intersection of (1) with (2) and (3). The conic is a circle and therefore the coefficient of xy in equation (4) must vanish.

This gives

$$lm'+ml'=0,$$

or

$$\frac{l}{m}=-\frac{l'}{m'}.$$

The slopes of (2) and (3) are thus equal in magnitude and opposite in sign showing that the common chords are equally inclined to the x-axis, the inclination being in opposite directions.

5. *Show that through four given points two parabolas can be drawn of which the axes are parallel to the asymptotes of the centre locus of the family of conics through the four points.*

Let us take the line joining two of the given points as the axis of x and the line joining the remaining two points as the axis of y.

Let $lx+my+1=0$ and $l'x+m'y+1=0$ be the equations of two straight lines which cut the coordinate axes in the four given points.

The equations

$$xy=0, (lx+my+1)(l'x+m'y+1)=0$$

represent two conics through the four points, and the equation

$$(lx+my+1)(l'x+m'y+1)+\lambda xy=0 \quad ...(1)$$

represents any conic passing through these points.

Writing equation (1) as

$$ll'x^2+(lm'+ml'+\lambda)xy+mm'y^2 +(l+l')x+(m+m')y+1=0, \quad ...(2)$$

we see that it represents a parabola if

$$4ll'mm'=(lm'+ml'+\lambda)^2. \quad ...(3)$$

Equation (3) gives two values of λ and consequently two parabolas will pass through the four given points.

When equation (2) represents a parabola, the second degree terms in it must form a perfect square so that they can be written as $(x\sqrt{ll'} \pm y\sqrt{mm'})^2$. The axes of the two parabolas will thus be parallel to the line $x\sqrt{ll'} \pm y\sqrt{mm'}=0$, or the line pair

$$ll'x^2-mm'y^2=0.$$

Now the centre of (1) is obtained from the equations

$$l\,(l'x+m'y+1)+l'\,(lx+my+1)+\lambda y=0,$$

and $$m\,(l'x+my'+1)+m'\,(lx+my+1)+\lambda x=0.$$

Eliminating λ, the locus of the centre is

$$2ll'x^2-2mm'y^2+(l+l')\,x-(m+m')\,y=0. \quad \ldots(4)$$

The asymptotes of the locus are parallel to the lines

$$ll'x^2-mm'y^2=0,$$

i.e., from (4), to the axes of the parabolas through the four given points.

14.5 Double contact. Let $S=0$ be a conic and $lx+my+n=0$, $l'x+m'y+n'=0$ be two straight lines.

The equation

$$S+\lambda\,(lx+my+n)\,(l'x+m'y+n')=0$$

represents a conic which passes through the four points of intersection of the conic $S=0$ and the given lines.

In particular,

$$S+\lambda\,(lx+my+n)^2=0 \quad \ldots(1)$$

is the equation of a conic which touches the conic $S=0$ at each of the two points in which the line $lx+my+n=0$ meets it. For the conic $(lx+my+n)^2=0$ meets $S=0$ in two pairs of coincident points.

We say that the conic represented by equation (1) has **double contact** with the conic $S=0$ at its points of intersection with line $lx+my+n=0$.

Note. In case the line $lx+my+n=0$ is a tangent to the conic $S=0$, the conic $S+\lambda\,(lx+my+n)^2=0$ will touch $S=0$ at four coincident points. We then say that $S+\lambda\,(lx+my+n)^2=0$ has *four point contact or contact of the third order with* $S=0$.

14.51 Pair of tangents. The property of double contact can be used to find the equation to the pair of tangents that can be drawn from a given point (x', y') to the conic $S=0$.

The equation to the chord of contact of tangents that can be drawn from (x', y') to $S=0$ is

$$T\equiv axx'+h\,(xy'+yx')+byy'+g\,(x+x')+f\,(y+y')+c=0 \quad \ldots(1)$$

and the equation of a conic having double contact with $S=0$ at the points where (1) meets it, is

$$S+\lambda T^2=0. \quad \ldots(2)$$

Now the pair of tangents is one such conic and will be obtained from equation (2) by choosing λ such that (2) passes through (x', y').

Substituting these coordinates in (2), we get

$$S'+\lambda S'^2=0,$$

i.e., $$\lambda=-\frac{1}{S'}.$$

Hence, from (2), the equation to the pair of tangents from (x', y') is

$$SS'=T^2.$$

14.52 Conic touching four fixed straight lines.

Let two of the straight lines be taken as coordinate axes and let $lx+my=1$, $l'x+m'y=1$ be the equations of the other two straight lines.

The equation of the conic touching the coordinate axes can be written as

$$(ax+by-1)^2+2\lambda xy=0. \quad \ldots(1)$$

The line joining the origin to the points in which the line $lx+my=1$ cuts (1) are given by the equation

$$(ax+by-lx-my)^2+2\lambda xy=0.$$

These lines are coincident if

$$(a-l)^2\,(b-m)^2=\{(a-l)\,(b-m)+\lambda\}^2,$$

i.e., if $$\lambda=-2\,(a-l)\,(b-m).$$

Similarly, the line $l'x+m'y=1$ touches the conic (1) if

$$\lambda=-2\,(a-l')\,(b-m').$$

Equation (1) therefore represents a conic touching the four fixed straight lines if

$$\lambda=-2\,(a-l)\,(b-m)=-2\,(a-l')\,(b-m').$$

Examples

1. Prove that the conics

$$x^2+3y^2-1=0,$$

and $$2x^2+12xy+39y^2-2x-12y=0,$$

have double contact with each other. Find the coordinates of the point of intersection of the tangents at the two points of contact.

(*Gorakhpur*, 1976; *Lucknow*, 1978)

Ans. (1, 2).

2. Find the equation of the parabola which touches the conic

$$x^2+xy+y^2-2x-2y+1=0$$

at the points where it is cut by the line $x+y+1=0$.

Determine the coordinates of the focus of this parabola.

(*Lucknow*, 1978; *Gorakhpur*, 1973)

Ans. $(x-y)^2=14(x+y)+1;\ \left(\frac{25}{14}, \frac{25}{14}\right)$.

3. Two conics $S_1=0$, $S=0$ have a pair of common chords $u=0$, $v=0$ such that $S_1-S_2\equiv uv$.

Prove that the

$$\lambda^2u^2-2\lambda(S_1+S_2)+v^2=0$$

has double contact with both $S_1=0$ and $S_2=0$. (*Lucknow*, 1960)

4. Prove that the locus of centres of conics which touch the coordinate axes at distances h and k from the origin is the straight line $hy-kx$. (*Lucknow*, 1979)

Hint. The conics have double contact with $xy=0$ at the extremities of the chord $\frac{x}{h}+\frac{y}{k}=1$.

5. Prove that the equation of the circle having double contact with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the ends of a latus rectum is

$$x^2+y^2-2ae^3x=a^2(1-e-e^4).$$

(*Gorakhpur*, 1969; *Lucknow*, 1980)

6. $S=0$ is the equation of a conic, $L=0$ is the equation of a line meeting the conic in two points P and Q, and $T=0$ is the equation of the tangent to the conic at P. Interpret the equations

(i) $S-\lambda L^2=0$, (ii) $S-\lambda LT=0$, (iii) $S-\lambda T^2=0$, where λ is a parameter.

A variable conic passes through two fixed points A, B and has double contact with a fixed conic. Prove that the chord of contact passes through one or other of two fixed point on AB.

7. If the chords of contact of two circles with a conic with which they have double contact be parallel, prove that the radical axis of the circles bisects the distance between these chords.

Solution. Taking the axes of the conic as coordinate axes, its equation can be written as $ax^2+by^2=1$.

The conic $ax^2+by^2-1+\lambda(lx+my+n)^2=0$ has double contact with the given conic. This is a circle if either $l=0$, or $m=0$, and $\lambda(l^2-m^2)=b-a$.

If $l=0$, the chords of contact are parallel to x-axis and if $m=0$, the chords of contact are parallel to y-axis.

Taking the second case, the equations of the two circles are

$$ax^2+by^2-1+(b-a)(x-k_1)^2=0$$

and

$$ax^2+by^2-1+(b-a)(x-k_2)^2=0.$$

The radical axis $2x-k_1-k_2=0$ of these circles bisects the distance between $x=k_1$, $x=k_2$.

8. The chords of contact of two circles with a conic with which they have double contact are perpendicular. Prove that the point of intersection of the chords is a limiting point belonging to the coaxal system determined by the two circles.

Solution. Let the conic be $ax^2+by^2=1$.

From the preceding example, the equations of the two circles are

$$S_1\equiv ax^2+by^2-1+(b-a)(x-k)^2=0$$

and

$$S_2\equiv ax^2+by^2-1-(b-a)(y-k')^2=0,$$

Now, $S_1-S_2\equiv(b-a)\{(x-k)^2+(y-k')^2\}=0$

is a circle belonging to the coaxal system determined by $S_1=0$, $S_2=0$, and, being a point circle, is the limiting point (k, k') which is also the point of intersection of the chords of contact.

9. Prove that the locus of a point, the sum of difference of the tangents from which two given circles is constant, is a conic having double contact with each of the two circles. (*Lucknow*, 1962)

Solution. Taking the line of centres as the x-axis and the radical axis as the y-axis, the equations of the two circles are

$$x^2+y^2+2g_1x+c=0,$$

and

$$x^2+y^2+2g_2x+c=0.$$

If (x, y) be the variable point, then

$$\sqrt{x^2+y^2+2g_1x+c}\pm\sqrt{x^2+y^2+2g_2x+c}=k,$$

where k is constant.

Taking the plus sign,

$$\sqrt{x^2+y^2+2g_1x+c}+\sqrt{x^2+y^2+2g_2x+c}=k; \quad \text{—(1)}$$

Also

$$(x^2+y^2+2g_1x+c)-(x^2+y^2+2g_2x+c)=2\,(g_1-g_2)\,x. \qquad \ldots(2)$$

From (1) and (2),

$$\sqrt{x^2+y^2+2g_1x+c}-\sqrt{x^2+y^2+2g_2x+c}=\frac{2\,(g_1-g_2)\,x}{k}. \qquad \ldots(3)$$

Adding (1) and (3) and squaring,

$$4\,(x^2+y^2+2g_1x+c)-\left\{\frac{2\,(g_1-g_2)}{k}\,x+k\right\}^2=0,$$

which has double contact with the first circle. Similarly, by subtracting (3) from (1), it can easily be seen that the locus of $(x,\ y)$ has double contact with the second circle.

If we take the difference of the tangents, the result follows in a similar manner.

10. Show that the centres of all conics which touch four fixed straight lines lie on a straight line.

14.6 Eccentricity of the conic $ax^2+2hxy+by^2=1$.

We know that the equation $ax^2+2hxy+by^2=1$ represents a conic of which the centre is at the origin. Let the axes of coordinates be so rotated as to coincide with the principal axes of the conic. If the transformed equation is $\alpha x^2+\beta y^2=1$, we have by invariants

$$\alpha+\beta=a+b.$$

and

$$\alpha\beta=ab-h^2.$$

Also, if e be the eccentricity, and $\alpha<\beta$, we have

$$e^2=\frac{\beta+\alpha}{\beta},$$

$$\therefore \quad \frac{e^2}{2-e^2}=\frac{\beta-\alpha}{\beta+\alpha}=\frac{\sqrt{(a+b)^2-4\,(ab-h^2)}}{a+b}$$

$$=\frac{\sqrt{(a-b)^2+4h^2}}{a+b}.$$

Squaring and simplifying,

$$e^4\,(ab-h^2)+\{(a-b)^2+4h^2\}\,(e^2-1)=0,$$

which gives the value of e.

14.61 Foci of central conics. Referred to its principal axes the equation of central conic is

$$\frac{x^2}{y^2}+\frac{y^2}{b^2}=1, \qquad \ldots(1)$$

which represents a hyperbola if b^2 is negative. The coordinates of the foci of (1) are

$$(\pm\sqrt{a^2-b^2},\ 0),$$

the corresponding directrices being

$$x=\pm\frac{a^2}{\sqrt{a^2-b^2}}.$$

From the symmetry of equation (1), it is apparent that there are two other foci with coordinates $(0,\ \pm\sqrt{(b^2-a^2)})$. These foci are imaginary since

$$b^2=a^2\,(1-e^2),$$

e being the eccentricity.

The corresponding directrices are

$$y=\pm\frac{b^2}{\sqrt{b^2-a^2}}.$$

It can be seen that each directrix is the polar of the corresponding focus.

A parabola being the limiting case of an ellipse, may also be regarded as having four foci, three of which lie at infinity.

We thus have the following result :

Every central conic has four foci, two real and two imaginary. The real foci lie on one axis of the conic and the imaginary ones on the other axis.

Ex. Show that every central conic has two eccentricities, of which one is real and the other imaginary if the conic is an ellipse, and both are real if the conic is a hyperbola.

14.62 Tangents from a focus. The equation to the pair of tangents from the focus $(\sqrt{a^2-b^2},\ 0)$ to the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{a^2-b^2}{a^2}-1\right)=\left(\frac{x\sqrt{(a^2-b^2)}}{a^2}-1\right)^2,$$

which on simplification gives

$$x^2-2x\sqrt{a^2+b^2}+(a^2-b^2)+y^2=0,$$

that is

$$(x-\sqrt{a^2-b^2})^2+y^2=0$$

which is a point circle at the focus $(\sqrt{a^2+b^2},\ 0)$ itself.

It can similarly be seen that the points of tangents from each of the three remaining foci are point circles at the corresponding foci.

14.63 Foci of a conic referred to rectangular axes.

Let $ax^2+2hxy+by^2+2gx+2fy+c=0$

be the equation to a conic when the axes are rectangular, and let (x', y') be the coordinates of a focus.

The equation to the pair of tangents from (x', y') is

$$(ax^2+2hxy+by^2+2gx+2fy+c)\ (ax'^2+2hx'y'+by'^2+2gx'+2fy'+c)$$
$$=\{x\ (ax'+hy'+g)+y\ (hx'+by'+f)+(gx'+fy'+c)\}^2.$$

Since this must reduce to a point circle at (x', y') the coefficients of x^2 and y^2 must be equal and the coefficients of xy must vanish.

This gives

$$(ax'^2+2hx'y'+by'^2+2gx'+2fy'+c)\ (a-b)$$
$$-(ax'+hy'+g)^2+(hx'+by'+f)^2=0,$$

and
$$h\ (ax'+2hx'y'+by'^2+2gx'+2fy'+c)$$
$$-(ax'^2+hy'+g)\ (hx'+by'+f)=0.$$

From these, we get

$$\frac{(\mathrm{ax}'+\mathrm{hy}'+\mathrm{g})^2-(\mathrm{hx}'+\mathrm{by}'+\mathrm{f})^2}{\mathrm{a}-\mathrm{b}}$$
$$=\frac{(\mathrm{ax}'+\mathrm{by}'+\mathrm{g})\ (\mathrm{hx}'+\mathrm{by}'+\mathrm{f})}{\mathrm{h}}.$$
$$=\mathrm{ax}'^2+2\mathrm{hx}'\mathrm{y}'+\mathrm{by}'^2+2\mathrm{gx}'+2\mathrm{fy}'+\mathrm{c},$$

an the equations to determine the coordinates of the foci of the general conic.

14.64 Axes and directrices of the conic $ax^2+2hxy+by^2+2gx+2fy...c=0$.

I. Axes. We have seen in § 14.61 that the foci of a conic lie upon its axes. We also know that the axes pass through the centre of the conic.

From the preceding article, the conic

$$\frac{(ax+hy+g)^2-(hx-by+f)^2}{a-b}=\frac{(ax+by+g)\ (hx+by+f)}{h} \quad ...(1)$$

passes through the foci of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \quad ...(2)$$

Now, since the coordinates of the centre of (2) satisfy the equations $ax+hy+g=0$, $hx-by+f=0$, the conic given by (1) also passes through the centre of (2).

Since the axes of (2) are the only conic passing through these five points (four foci and centre), no four of which are collinear, equation (1) represents the axes of the given conic.

II. Directrices. Since the directrices of a conic are the polars of, or which is the same as the chords of contact of tangents from the corresponding foci, the equation of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is equivalent to the equation

$$(x-x')^2+(y-y')^2-(lx+my+n)^2=0,$$

where $lx+my+n=0$ is the equation of the directrix corresponding to the focus (x', y').

Comparing coefficients,

$$\frac{l^2-1}{a}=\frac{lm}{h}=\frac{m^2-1}{b}=\frac{ln+x'}{g}$$
$$=\frac{mn+y'}{f}=\frac{n^2-x'^2-y'^2}{c}=\lambda, \text{ say.}$$

We then have

$$\lambda\ (ax'+hy'+g)=l\ (lx'+my'+n), \quad ...(1)$$
$$\lambda\ (hx'+by'+f)=m\ (lx'+my'+n), \quad ...(2)$$
$$\lambda\ (gx'+fy'+c)=n\ (lx'+my'+n), \quad ...(3)$$
$$\lambda\ (a-b)=l^2-m^2, \quad ...(4)$$
$$\lambda h=lm. \quad ...(5)$$

Eliminating x', y' from (1), (2) and (3), we get

$$\begin{vmatrix} \lambda a-l^2 & \lambda h-lm & \lambda g-ln \\ \lambda h-lm & \lambda b-m^2 & \lambda f-mn \\ \lambda g-ln & \lambda f-mn & \lambda c-n^2 \end{vmatrix}=0,$$

as the cubic to determine λ.

Two roots of the above cubic in λ are zero. The third root when substituted in (4) and (5) gives the ratio $l : m : n$ which determine the directrices.

Examples

1. Determine the foci and directrices of the conic

$$x^2+12xy-4y^2-6x+4y+9=0.$$

Ans. Real foci $(1, -1)$, $(-1, 2)$;
Real directrices $2x-3y+4=0$,
$2x-3y-1=0$.

2. Show that the general equation of conics whose foci are the points (α, β) and $(-\alpha, -\beta)$ may be expressed as

$$(x^2+\beta^2-\lambda)(y^2+a^2-\lambda)=(xy-\alpha\beta)^2.$$

14.7 Circular points at infinity. From Chapter III, we know that the equation $c=0$ represents a straight line lying wholly at infinity. We shall now study the following foci associated with the line at infinity.

1. $S=\lambda u$; $S=0$, $u=0$ *being a conic and a line respectively and* λ *a constant.*

Since $\lambda=0$ is the equation of the line at infinity, $S=\lambda u$ represents a conic passing through the points where the conic $S=0$ is cut by $u=0$ and the line at infinity. The two conics thus have the same intersections with the line at infinity. Their asymptotes are parallel.

In particular, take the circle

$$x^2+y^2-2gx-2fy-c=0. \qquad ...(1)$$

Equation (1) can be written as

$$x^2+y^2-a^2=2gx+2fy+c-a^2. \qquad ...(2)$$

Now, equation (2) has the form $S=\lambda u$.

The circles

$$x^2+y^2-a^2=0,$$

and $$x^2+y^2-2gx-2fy-c=0,$$

therefore meet the line at infinity in the same two (imaginary) points. These are called the **circular points at infinity.**

2. $S=\lambda$.

Writing the equation as $S=\lambda\ (0.x+0.y+1)^2$, we see that the conics $S=0$ and $S=\lambda$ have double contact at the points where they meet the line at infinity. It is easy to see that the two conics have the same pair of real or imaginary asymptotes.

In particular, if $S=0$ is a circle, $S=\lambda$ is a concentric circle and the two have double contact at the circular points at infinity.

14.8 Solved examples.

1. Given the focus and directrix of a conic, show that polar of a given point with respect to it passes through a fixed point.

Solution. Take the coordinates of focus as (0, 0) and the equation of the directrix as $x=a$. The equation of the conic is then

$$x^2+y^2=e^2(x-a)^2.$$

The polar of any point (x_1, y_1) with respect to the above conic is

$$xx_1+yy_1=e^2[xx_1-a(x+x_1)+a^2].$$

This passes through the point of intersection of $xx_1+yy_1=0$ and $xx_1-a(x+x_1)+a^2=0$, which is a fixed point.

2. A conic has double contact with the parabola $y^2=4ax$. If the chord of contact passes through the vertex of the parabola and the conic passes through its focus, prove that the locus of the centre of the conic is the parabola $y^2=a(2x\ \ a)$.

(*Gorakhpur*, 1966; *Lucknow*, 1979)

Solution. Any line through the vertex of the parabola $y^2=4ax$ is $y=mx$. A conic having double conic with the parabola is

$$y^2-4ax+\lambda(mx-y)^2=0$$

or $$\lambda m^2x^2-2\lambda mxy+y^2(1+\lambda)-4ax=0. \qquad ...(1)$$

The coordinates of centre satisfy the equation

$$\lambda m^2x-\lambda my-2a=0$$

and $$-\lambda mx+y(1+\lambda)=0.$$

These give $$x=\frac{2a(1+\lambda)}{\lambda m^2},\ y=\frac{2a}{m}.$$

Since (1) passes through the focus $(a, 0)$ of the parabola,

$$\lambda m^2=4.$$

Elimination of λ between these three equations gives the required locus of the centres of the conic of the family.

3. A parabola of given latus rectum touches a fixed equal parabola, the axes of the two curves being parallel. Prove that the vertex of the moving parabola lies on another parabola whose latus rectum is double that of the fixed parabola.

Solution. Let the equation of the fixed parabola be

$$y^2=4ax. \qquad ...(1)$$

The equation of another moving parabola of equal latus rectum and axes parallel to the given parabola would be

$$(y-k)^2+4a(x-h)=0. \qquad ...(2)$$

The points of intersection of (1) and (2) are given by

$$2y^2-2ky+k^2-4ah=0.$$

This equation should give two equal values of y since the two parabolas touch each other. Therefore

$$k^2=8ah.$$

Hence, the locus of (h, k), the vertex of (2) is $y^2=8ax$, a parabola having latus rectum double that of the fixed parabola.

4. The line $x+y+1=0$ forms with the line pair $ax^2+2hxy+by^2=0$, a triangle of which the vertex A lies at the origin. Show that

the rectangular hyperbola

$$2(ax^2+2hxy+by^2)-(a+b)(x+y+1)^2=0$$

touches AB at B and AC at C.

Show further that the equation of the circumcircle of the triangle ABC is

$$(a+b-2h)(x^2+y^2)+(b-a-2h)x+(a-b-2h)y=0.$$

Solution. Any conic which has double contact with $ax^2+2hxy+by^2=0$ at the extremities of the line $x+y+1=0$ is

$$ax^2+2hxy+by^2+\lambda(x+y+1)^2=0. \qquad \text{...(1)}$$

This will represent a rectangular hyperbola when

$$\lambda=-\tfrac{1}{2}(a+b).$$

Putting this value of λ in (1), we get the required equation of the rectangular hyperbola satisfying the given conditions.

Further, let the equation of the circumscribing circle passing through the origin be

$$x^2+y^2+2gx+2fy=0. \qquad \text{...(2)}$$

The equation of pair of straight lines joining the origin to the points of intersection of the above circle and the line $x+y+1=0$ is

$$(x^2+y^2)-(2gx+2fy)(x+y)=0,$$

or

$$x^2(1-2g)+y^2(1-2f)-2xy(g+f)=0. \qquad \text{...(3)}$$

This equation is the same as

$$ax^2+2hxy+by^2=0. \qquad \text{...(4)}$$

Comparing coefficients in (3) and (4), we obtain

$$\frac{1-2g}{g}=\frac{-(g+f)}{h}=\frac{1-2f}{b},$$

so that

$$2f=\frac{a-b-2h}{a+b-2h} \text{ and } 2g=\frac{b-a-2h}{b+a-2h}.$$

Putting these values of g and f in (2), the equation of the circumcircle of the triangle ABC is obtained.

Examples on Chapter XIV

1. A conic passes through the five points $(a, 0)$, $(b, 0)$, $(0, c)$, $(0, d)$ and (b, d), where a, b, c, d are positive and $b>a$, $d>c$. Show that it is an ellipse and that if $b=2a$, $d=2c$, the centre is the point $\left(\frac{6a}{5}, \frac{6c}{5}\right)$.

(*Lucknow*, 1979)

2. A circle cuts the parabola $y^2=4ax$ in four points 't_1', 't_2', 't_3', and 't_4'. Show that $t_1+t_2+t_3+t_4=0$.

3. For what values of h, if any, does the equation

$$x^2+2hxy+4y^2+6x+12y=0$$

represent (i) a pair of straight lines, (ii) a parabola, (iii) an ellipse, and (iv) a hyperbola ?

Show that in cases (iii) and (iv), the centres lie on the line $x-2y=0$.

4. A circle and a rectangular hyperbola intersect in four points and one of their common chords is a diameter of the hyperbola. Show that the other chord is a diameter of the circle.

(*Lucknow*, 1980)

5. Show that the conic

$$c(x^2+y^2)+2xy\sqrt{(a-c)(b-c)}=1$$

has double contact with both the conics

$$ax^2+by^2=1 \text{ and } bx^2+ay^2=1. \qquad \text{(I. A. S., 1966)}$$

6. Prove that there will be two circles passing through the origin which have double contact with the conic

$$(x^2-y^2)\cos 2\alpha+2xy\sin 2\alpha+2=0. \qquad \text{(Lucknow, 1979)}$$

7. A and B are two fixed points and P a variable point. The angle PAB is θ and angle PAB is ϕ. Prove that :

(i) If $a\tan\theta+b\tan\phi=c$, the locus of P is, in general, a hyperbola passing through A and B.

(ii) if $a\tan\theta+b\tan\phi=0$, the locus of P is a straight line perpendicular to AB.

(iii) if $\tan\theta+\tan\phi=k$, the locus of P is a parabola whose axis is perpendicular to AB.

(vi) if $\sin\theta=\mu\sin\phi$, the locus of P is a circle.

8. A system of conic have their principal axes along two straight lines and they all pass through a given point. Prove that the poles of a given line with respect to the system of conic lie on a rectangular hyperbola.

9. A rectangular hyperbola has double contact with a given parabola. Prove that the centre of the hyperbola and the pole of the chord of contact will be equidistant from the directrix of the parabola.

10. A rectangular hyperbola has double contact with a fixed central conic. If the chord of contact always passes through a fixed point, prove that the locus of the centre of the rectangular hyperbola is a circle passing through the centre of the fixed conic.

(*Gorakhpur*, 1971; *Lucknow*, 1980)

11. Prove that the locus of poles of PQ with respect to all conics passing through four fixed points P, Q, R, is a straight line.

12. Prove that, in general two parabolas and one rectangular hyperbola can be drawn through the intersection of two conics, and

that the asymptotes of the rectangular hyperbola are parallel to the bisectors of the angles between the axes of the two parabolas.

(*Ranchi*, 1968)

13. Prove that the locus of the vertices of the rectangular hyperbola $x^2-y^2 \ldots 2\lambda xy-a^2=0$ for different values of λ is the curve whose equation is $(x^2+y^2)^2-a^2(x^2-y^2)=0$.

14. Two conics have double contact with each other. Prove that the polar of centre of one conic with respect to the other is parallel to the chord of contact. (*Lucknow*, 1980; *Gorakhpur*, 1977)

15. Two circles each have double contact with a parabola and touch each other. Prove that the difference between their radii is equal to the latus rectum of the parabola.

16. A family of conics have double contact with a given conic at the extremities of a given chord. Show that the locus of the centres of conics of the family is the diameter of the given conic conjugate to the given chord.

17. A circle has double contact with a hyperbola. If the chord of contact be parallel to the transverse axis, prove that the ratio of the length of the tangent to the circle from any point of the hyperbola to the distance of the point from the chord of contact is equal to the eccentricity of the conjugate hyperbola.

18. A circle of given radius cuts an ellipse in four points, prove that the continued product of the diameters of the ellipse parallel to the common chords is constant.

19. Prove that if a conic circumscribe a quadrilateral, the ratio of the product of the perpendiculars from any point P of the conic upon two opposite sides of the quadrilateral to the product of the perpendiculars from P upon the other two sides is the same for all positions of P.

20. Show that the locus of the centres of rectangular hyperbolas which have four point contact with a given parabola is an equal parabola having the same axis and directrix.

21. Prove that the locus of centres of conics having four point contact with $x^2/a^2+y^2/b^2=1$ at $(a\cos\theta, b\sin\theta)$ is the line

$$bx\sin\theta=ay\cos\theta.$$

22. A circle is drawn to touch the parabola $y^2=4ax$ at the point P and to cut it in the origin and in the further point Q. Prove that PQ touches the parabola $y^2+32ax=0$.

23. With a fixed point O for centre any circle is described cutting a conic in points, real and imaginary. Show that the locus of the centre of all conics through these four points is a rectangular hyperbola, which is independent of the radius of the circle.

24. The normals to a central conic at four points P, Q, R and S meet in a point and the circle through the points P, Q, R cuts the conic again in S'. Show that SS' is a diameter of the conic.

25. Show that the envelop of a line L which meets a given circle at a varying point P such that the angle between L and the join of P to another fixed point A is 90 degrees, is a conic.

(*I. A. S.*, 1974)

Hint. Take the fixed circle as $x^2+y^2=a^2$ and the fixed point A as (h, k).

26. A circle and an ellipse have double contact with one another. Prove that the length of tangent drawn from any point of the ellipse to the circle varies as the distance of that point from the chord of contact.

27. The polars of the point $P_1:(x_1, y_1)$, $P_2:(x_2, y_2)$ with respect to the ellipse $x^2/a^2+y^2/b^2=1$ meet the curve in four points. Interpret the equation

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1\right)$$
$$-\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}-1\right)=0.$$

CHAPTER XV.

CIRCLE OF CURVATURE, CONFOCAL CONICS

15.1 Circle of curvature. Let (x', y') be a point on the conic

$$S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0. \qquad ...(1)$$

The equation of tangent at (x', y') to $S=0$ is

$$T \equiv axx'+h(xy'+yx')+byy'+g(x+x')+f(y+y')+c=0.$$

Now, $$S+\lambda T(lx+my+n)=0 \qquad ...(2)$$

is the equation of a conic which meets $S=0$ in points where $T=0$ and the line $$lx+my+n=0 \qquad ...(3)$$

meet it.

But $T=0$ meets the conic $S=0$ in two coincident points.

The conics (1) and (2) therefore touch each other at (x', y') which constitutes a pair of coincident points for both of them.

If (3) also passes through (x', y'), it has the form

$$l(x-x')+m(y-y')=0,$$

and equation (2) which then becomes

$$S+\lambda T\{l(x-x')+m(y-y')\}=0 \qquad ...(4)$$

represents a conic having three point contact at (x', y') with $S=0$.

If the coefficients in (4) are so chosen that it is the equation of a circle, then this circle is called the **circle of curvature,** or the **osculating circle** of the conic $S=0$ at the point (x', y').

Note. The circle of curvature of a conic at a given point has *three point contact of the second order* with the conic at the point. This is the limiting case in which two adjacent chords of intersection of a circle and a conic coincide with the tangent at the point, that is, the circle and the conic have two consecutive tangents in common.

Defining curvature as the rate of deflection of the tangent at any point of a curve, we see that **curvature** of a conic at any point is equal to the reciprocal of the radius of the circle of curvature, or as is technically termed, the radius of curvature of the conic at that point.

The **centre of curvature** at any point of a curve is the centre of the circle of curvature at the point.

Examples

1. *Find the equation of the circle of curvature and the coordinates of the centre of the curvature at the point* $(at^2, 2at)$ *of the parabola* $y^2=4ax$.

Show also that the circle of curvature meets the parabola again in the point $(9at^2, -6at)$.

Solution. The equation of the tangent at $(at^2, 2at)$ is

$$ty=x+at^2,$$

or $$y-\frac{x}{t}-at=0.$$

Since the chords of intersection of a conic and a circle, taken in pairs, are equally inclined to the axes of the conic, the equation to the common chord of the parabola and its circle of curvature at $(at^2, 2at)$ will be

$$y+\frac{x}{t}=2at+\frac{at^2}{t}=3at.$$

The circle of curvature will thus be

$$y^2-4ax+\lambda\left(y-\frac{x}{t}-at\right)\left(y+\frac{x}{t}=3at\right)=0,$$

provided λ is so chosen that the coefficients of x^2 and y^2 become equal to each other.

This gives $$1+\lambda=-\frac{\lambda}{t^2}, \text{ i.e., } \lambda=-\frac{t^2}{1+t^2}.$$

The equation of the circle of curvature is therefore

$$y^2-4ax-\frac{t^2}{1+t^2}\left(y-\frac{x}{t}-at\right)\left(y+\frac{x}{t}-3at\right)=0,$$

that is

$$(1+t^2)(y^2-4ax)-t^2\left(y^2-\frac{x^2}{t^2}-4aty+2ax+3a^2t^2\right)=0,$$

or $$x^2+y^2-2ax(2+3t^2)+4at^3y-3a^2t^4=0.$$

From this equation we easily see that coordinates of the centre of curvature are

$$\{a(2+3t^2), -2at^3\},$$

and the radius of curvature is

$$2at(1+t^2)^{3/2}.$$

Next, let the circle of curvature meet the parabola again in the point $(a't^2, 2at')$.

Then $$y+\frac{x}{t}=3at$$

passes through $(at'^2, 2at')$.

Therefore $$2at'+\frac{at'^2}{t}=3at$$

or $$3t^2-2tt'-t'^2=0,$$

from which $t'=t$, or $t'=-3t$.

The other point of intersection is therefore $(9at^2, -6at)$.

2. *Show that the radius of curvature at a pointe* $P:(a\cos\phi, b\sin\phi)$ *of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is $\dfrac{CD^2}{ab}$ *where CD is the semi-diameter conjugate to CP.*

If θ be the eccentric angle of the point in which the circle of curvature again cuts the ellipse, show that

$$\phi=\frac{1}{3}(2n\pi-\theta).$$

Solution. The tangent at P is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1.$$

The common chord of the ellipse and its circle of curvature at P is

$$\frac{x}{a}\cos\phi-\frac{y}{b}\sin\phi=\cos^2\phi-\sin^2\phi=\cos 2\phi. \quad ...(1)$$

The first part is worked out as in Example 1.

For the second part, we see that the point $(a\cos\theta, b\sin\theta)$ lies on (1). Therefore,

$$\cos\theta\cos\phi-\sin\theta\sin\phi=\cos 2\phi,$$

i.e., $$\cos(\theta+\phi)-\cos 2\phi=0,$$

or $$\sin\frac{\theta+3\phi}{2}\sin\frac{\phi-\theta}{2}=0.$$

Since $\sin\dfrac{\phi-\theta}{2}\neq 0$, $\dfrac{\theta+3\phi}{2}=n\pi$, where n is an integer or zero.

Hence, $$\phi=\frac{1}{3}(2n\pi-\theta).$$

3. *The circles of curvature of a fixed parabola at the extremities of a focal chord meet the parabola again at H and K. Prove that HK passes through a fixed point.*

Solution. Let the equation of the parabola be $y^2=4ax$. If 't_1', 't_2' be the extremities of a focal chord, then from example 1, we see that the coordinates of H, K are $(9at_1^2, -6at_1)$, $(9at_2^2, -6at_2)$.

Also $$t_1t_2=-1.$$

The equation to HK is

$$y+6at_1=\frac{6a(t_2-t_1)}{9a(t_1^2-t_2^2)}(x-9at_1^2),$$

or $$3y(t_1+t_2)+2x-18at_1t_2=0$$

i.e., $$3y(t_1+t_2)+2x+18a=0.$$

This always passes through the point $(0, -9a)$.

4. If the circles of curvature at three points of an ellipse all cut the ellipse again in the same point show that the centre of the ellipse is the centre of mean of the three points.

Hint. Use the result of Example 2, *viz.*, that the circles of curvature at the three points whose eccentric angles are $\frac{1}{3}(2\pi-\theta)$, $\frac{1}{3}(4\pi-\theta)$, $\frac{1}{3}(6\pi-\theta)$ cut the ellipse again in the same point whose eccetric angle is θ.

5. If δ and θ be the radii of curvature at the ends P and D of semi-conjugate diameters of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, then prove that

$$\delta^{2/3}+\theta^{2/3}=\frac{a^2+b^2}{a^{2/3}b^{2/3}}.$$

Prove further that the locus of the middle point of the line joining the centres of curvature at P and D is the curve

$$(ax+by)^{2/3}+(ax-by)^{2/3}=(a^2-b)^{2/3}.$$

6. Any tangent drawn to the circle of the curvature at the vertex of the parabola $y^2=4ax$ meets the parabola in points whose ordinates are y_1 and y_2. Show that

$$\frac{1}{y^1}\sim\frac{1}{y_2}=\frac{1}{2a}$$

15.2 Confocal conics. Conics having the same foci are said to form a confocal system.

In the preceding chapter, we have seen that the foci of a central conic lie on its principal axis.

All confocal central conics will therfore have a common centre and common principal axes.

If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the equation of an ellipse, the general equation of all conics confocal with it will be of the form

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1. \qquad \text{...(1)}$$

Since the distance of a focus from the centre is the same for all confocals, we have

$$a_1^2-b_1^2=a^2-b^2=\text{constant}.$$

This gives

$$a_1^2-a^2=b_1^2-b^2=\lambda, \text{ say}$$

where λ is a constant.

The general equation (1) now becomes

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

which gives different conics confocal with the ellipse for different values of λ.

15.21 Confocals through a given point. Let

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \qquad \text{...(1)}$$

be a conic confocal with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let (x_1, y_1) be the given point.

Since (1) passes through (x_1, y_1),

$$\frac{x_1^2}{a^2+\lambda}+\frac{y_1^2}{b^2+\lambda}=1. \qquad \text{...(2)}$$

Now, (2) is an equation in λ. Writing it as

$$f(\lambda)\equiv(a^2+\lambda)(b^2+\lambda)-x_1^2(b^2+\lambda)-y_1^2(a^2+\lambda)=0,$$

and taking the case when $a>b$, we see that $f(\lambda)$ is $+ve$ for $\lambda=+\infty$, $-ve$ for $\lambda=-b^2$, and again $+ve$ for $\lambda=-a^2$.

The equation in λ has therefore both roots real, one lying between $+\infty$ and $-b^2$, and the other between $-b^2$ and $-a^2$.

For the root lying between $+\infty$ and $-b^2$, both $a^2+\lambda$ and $b^2+\lambda$ are positive, and for the root lying between $-b^2$ and $-a^2$, $a^2+\lambda$ is $+ve$ but $b^2+\lambda$ is $-ve$.

We thus have the following result :

Through any point in the plane of an ellipse, there passes two confocals, one of which is an ellipse and the other hyperbola.

15.22 Coordinates of a point in terms of the parameters of the confocals through it.

Let (x_1, y_1) be a point in the plane of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and let λ_1, λ_2 be the parameters of the confocals through (x_1, y_1). We then have

$$\frac{x_1^2}{a^2+\lambda_1}+\frac{y_1^2}{b^2+\lambda_1}=1,$$

and

$$\frac{x_1^2}{a^2+\lambda_2}+\frac{y_1^2}{b^2+\lambda_2}=1.$$

From these,

$$\frac{x_1^2}{\frac{1}{b^2+\lambda_2}-\frac{1}{b^2+\lambda_1}}=\frac{y_1^2}{\frac{1}{a^2+\lambda_1}-\frac{1}{a^2+\lambda_2}}$$

$$=\frac{1}{\frac{1}{(a^2+\lambda_1)(b^2+\lambda_2)}-\frac{1}{(a^2+\lambda_2)(b^2+\lambda_1)}}$$

$$=\frac{(a^2+\lambda_1)(a^2+\lambda_2)(b^2+\lambda_1)(b^2+\lambda_2)}{(\lambda_1-\lambda_2)(a^2-b^2)}$$

Therefore,

$$x_1^2=\frac{(a^2+\lambda_1)(a^2+\lambda_2)}{a^2-b^2},$$

$$y_1^2=\frac{(b^2+\lambda_1)(b^2+\lambda_2)}{b^2-a^2}.$$

15.3 Confocal parabolas. Parabolas having a common focus and common axis are said to be confocal. If the common focus is taken as the origin and the common axis, the axis of x, the equation

$$y^2=4a(x+a)$$

will give a system of confocal parabola for different values of a.

15.4 Proposition on confocals.

(*i*) *The confocals through any point in the plane of ellipse intersect orthogonally.*

(*Rajasthan*, '7[illegible], *Gorakhpur*, 1976; *Lucknow*, 1980)

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and let the two confocal through (x_1, y_1) be

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \text{ and } \frac{x^2}{b^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1.$$

Since (x_1, y_1) lies on both,

$$\frac{x_1^2}{a^2+\lambda_1}+\frac{y_1^2}{b^2+\lambda_1}=1, \qquad \text{...(1)}$$

$$\frac{x_1^2}{a^2+\lambda_2}+\frac{y_1^2}{b^2+\lambda_2}=1. \qquad \text{...(2)}$$

Subtracting (2) from (1) and removing the common factor $\lambda_2-\lambda_1$, we obtain

$$\frac{x_1^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y_1^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0.$$

This is also the condition that the tangents

$$\frac{xx_1}{a^2+\lambda_1}+\frac{yy_1}{b^2+\lambda_1}=1,$$

and

$$\frac{xx_1}{a^2+\lambda_2}+\frac{yy_1}{b^2+\lambda_2}=1,$$

to the two confocals at their common point (x_1, y_1) be at right angles.

This proves the proposition.

(*iii*) *One conic and only one of a confocal system will touch a given straight line.*

Let the given straight line be

$$x \cos \alpha+y \sin \alpha=p. \qquad \text{...(1)}$$

and let the confocal system be given by

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1. \qquad \text{...(2)}$$

The line (1) touches (2) if

$$(a^2+\lambda) \cos^2 \alpha+(b^2+\lambda) \sin^2 \alpha=p^2.$$

This gives only one value of λ. Hence the proposition is proved.

(*iii*) *The point of intersection of two perpendicular tangents one to each of two given confocals lies on a circle.*

(*Gorakhpur*, 1972; *Lucknow*, 1979)

Let the given confocals be

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1, \qquad \text{...(1)}$$

and

$$\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1. \qquad \text{...(2)}$$

The perpendicular straight lines

$$x \cos \alpha+y \sin \alpha=\sqrt{\{(a^2+\lambda_1) \cos^2 \alpha+(b^2+\lambda_1) \sin^2 \alpha\}}$$

and

$$x \sin \alpha-y \cos \alpha=\sqrt{\{(a^2+\lambda_2) \sin^2 \alpha+(b^2+\lambda_2) \cos^2 \alpha\}}$$

touch the conics (1) and (2) respectively.

Squaring and adding, the locus of the point of intersection of the above tangents is

$$x^2+y^2=a^2+b^2+\lambda_1+\lambda_2,$$

which is a circle.

(*iv*) *If two conics confocal with a given conic* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *pass through a given point* P, *and* PQ, PR *the normals at* P *to the confocals meet polar of* P *with respect to the given conic* Q *and* R, *then*

$$PQ=-\frac{\lambda_1}{p_1},\ PR=-\frac{\lambda_2}{p_2},$$

where p_1, p_2 *are perpendiculars from the centre to the tangents at* P *to the confocals, and* λ_1, λ_2 *the parameters of the confocals.*

Let the coordinates of P be (α, β).

The equation to the tangent at P to the confocal

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1, \qquad \text{...(1)}$$

is

$$\frac{x\alpha}{a^2+\lambda_1}+\frac{y\beta}{b^2+\lambda_1}=1. \qquad \text{...(2)}$$

The equation of the normal to (1) at P is

$$\frac{\beta}{b^2+\lambda_1}(x-\alpha)-\frac{\alpha}{a^2+\lambda_1}(y-\beta)=0,$$

which can be written as

$$\frac{x-\alpha}{\frac{p_1\alpha}{a^2+\lambda_1}}=\frac{y-\beta}{\frac{p_1\beta}{b^2+\lambda_1}}=r$$

where

$$p_1=\frac{1}{\sqrt{\frac{\alpha^2}{(a^2+\lambda_1)^2}+\frac{\beta^2}{(b^2+\lambda_1)^2}}}$$

which represents the length of the perpendicular from the centre upon the tangent to (1), and r is the distance of (x, y) from (α, β). Hence, if $PQ=r$, the coordinates of Q are

$$\alpha\left(2+\frac{p_1r}{a^2+\lambda_1}\right),\ \beta\left(1+\frac{p_1r}{b^2+\lambda_1}\right).$$

But Q is on the polar of P, and therefore

$$\frac{\alpha^2}{a^2}\left(1+\frac{p_1r}{a^2+\lambda_1}\right)+\frac{\beta^2}{b^2}\left(1+\frac{p_1r}{b^2+\lambda_1}\right)$$

$$=1=\frac{a^2}{a^2+\lambda_1}+\frac{\beta^2}{b^2+\lambda_1},$$

since P lies on (2).

This gives

$$(p_1r+\lambda_1)\left\{\frac{\alpha^2}{a^2(a^2+\lambda_1)}+\frac{\beta^2}{b^2(b^2+\lambda_1)}\right\}=0.$$

Therefore,

$$PQ=r=-\frac{\lambda_1}{p_1}.$$

Similarly,

$$PR=-\frac{\lambda_2}{p_2}.$$

Examples

1. Find the conics confocal with $x^2+2y^2=2$ which pass through the point (1, 1). (*Lucknow*, 1978)

Ans. $3x^2-y^2\pm\sqrt{5}\,(x^2-y^2)=2$.

2. Show that the confocal hyperbola through the point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentric angle is θ, is

$$\frac{x^2}{\cos\theta}-\frac{y^2}{\sin^2\theta}=a^2-b^2. \qquad (Gorakhpur,\ 1975)$$

Solution. Let the confocal through $(a\cos\theta, b\sin\theta)$ be

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

Then,

$$\frac{a^2\cos^2\theta}{a^2+\lambda}+\frac{b^2\sin^2\theta}{b^2+\lambda}=1.$$

Arranging the above equation in powers of λ, we have

$$\lambda^2+\lambda\,(a^2\sin^2\theta+b^2\cos^2\theta)=0,$$

from which,

$$\lambda=0, \text{ or } -(a^2\sin^2\theta+b^2\cos^2\theta).$$

$\lambda=0$ gives the ellipse itself. The equation to the confocal hyperbola is therefore

$$\frac{x^2}{a^2-(a^2\sin^2\theta+b^2\cos^2\theta)}+\frac{y^2}{b^2-(a^2\sin^2\theta+b^2\cos^2\theta)}=1,$$

i.e.,

$$\frac{x^2}{\cos^2\theta}-\frac{y^2}{\sin^2\theta}=a^2-b^2.$$

3. Show that the locus of the pole of a given straight line with respect to a system of confocal conics is a straight line. (*Gorakhpur*, 1971; *Lucknow*, 1979)

4. If θ be the angle between the tangents from a given point P to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

show that

$$\tan\theta=\frac{2\sqrt{-\lambda_1\lambda_2}}{\lambda_1+\lambda_2}$$

where λ_1, λ_2 are the parameters of the confocals through P.

Solution. Let P be the point (x_1, y_1), and let the confocal through P be

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

Since this passes through (x_1, y_1),

$$\frac{x_1^2}{a^2+\lambda}+\frac{y_1^2}{b^2+\lambda}=1,$$

i.e., $$\lambda^2+\lambda\,(a^2+b^2-x_1^2-y_1^2)+a^2b^2\left(1-\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}\right)=0. \qquad \ldots(1)$$

Now, the equation to the pair of tangents from (x_1, y_1) to the given ellipse is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

Therefore,

$$\tan\theta=\frac{2ab\sqrt{\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)}}{x_1^2+y_1^2-a^2-b^2}. \qquad \ldots(2)$$

If λ_1, λ_2 be the roots of (1),

$$\lambda_1+\lambda_2=x_1^2+y_1^2-a^2-b^2,$$

$$\lambda_1\lambda_2=a^2b^2\left(1-\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}\right).$$

Substituting in (2),

$$\tan\theta=\frac{2\sqrt{-\lambda_1\lambda_2}}{\lambda_1+\lambda_2}.$$

5. Prove that the two conics

$$ax^2+2hxy+by^2=1 \text{ and } a'x^2+2h'xy+b'y^2=1$$

can be placed so as to be confocal, if

$$\frac{(a-b)^2+4h^2}{(ab-h^2)^2}=\frac{(a'-b')^2+4h'^2}{(a'b'-h'^2)^2}.$$

Solution. Let the given conics referred to their principal axes be respectively

$$\alpha x^2+\beta y^2=1, \text{ and } \alpha' x^2+\beta' y^2=1.$$

If the conics can be placed so to be confocal,

$$\frac{1}{\alpha}-\frac{1}{\beta}=\frac{1}{\alpha'}-\frac{1}{\beta'},$$

i.e.,

$$\frac{\alpha-\beta}{\alpha\beta}=\frac{\alpha'-\beta'}{\alpha'\beta'}. \qquad \text{...(1)}$$

Now, by invariants,

$$\left.\begin{aligned}&\alpha+\beta=a+b,\ \alpha\beta=ab-h^2\\ \text{and}\quad &\alpha'+\beta'=a'+b',\ \alpha'\beta'=a'b'-h'^2.\end{aligned}\right\} \qquad \text{...(2)}$$

From (1),

$$\frac{(\alpha+\beta)^2-4\alpha\beta}{\alpha^2\beta^2}=\frac{(\alpha'+\beta')^2-4\alpha'\beta'}{\alpha'^2\beta'^2}.$$

Substituting in the above relation from (2), we have

$$\frac{(a+b)^2-4(ab-h^2)}{(ab-h^2)^2}=\frac{(a'-b')^2-4(a'b'-h'^2)}{(a'b'-h'^2)^2},$$

i.e.,

$$\frac{(a-b)^2+4h^2}{(ab-h^2)^2}=\frac{(a'-b')^2+4h'^2}{(a'b'-h'^2)^2}.$$

Examples on Chapter XV

1. Prove that the difference of the squares of the perpendiculars drawn from the centre on any two parallel tangents to two given confocals, is constant. (*Rajasthan*, 1971; *Gorakhpur*, 1976)

2. If the confocals through (x_1, y_1) to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \text{ and } \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$$

show that

(i) $\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1=-\dfrac{\lambda_1\lambda_2}{a^2b^2}$,

(ii) $x_1^2+y_1^2-a^2-b^2=\lambda_1+\lambda_2$.

3. Two conics S_1, S_2 have a common centre (x', y'); S_1 touches the x-axis at the origin, S_2 touches the y-axis at the origin. Write down suitable equations for S_1, S_2 and find the condition that they be confocals.

Ans. $S_1\equiv a'y'x^2-2a'x'xy+b'y'y^2-2(b'y'^2-a'x'^2)\,y=0.$

$S_2\equiv ax'x^2-2by'xy+bx'y^2-2(ax'^2-by'^2)=0.$

$$\left(\frac{a-b}{a'-b'}\right)^2=\frac{b^2y'^4}{a'^2x'^4}.$$

4. If λ, μ be the parameters of the confocals through two points P, Q on a given ellipse, show that (i) if P, Q be the extremities of conjugate diameters, then $\lambda+\mu$ is constant and (ii) if the tangents at P and Q be at right angles, then $\dfrac{1}{\lambda}+\dfrac{1}{\mu}$ is constant. (*1st Part : Gorakhpur*, 1973)

5. Tangents are drawn to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

from any point T on a given hyperbola confocal with the ellipse; if 2θ be the angle between the tangents, prove that $\sin\theta$ varies inversely as CD, where CD is the semi-diameter conjugate to CT of the ellipse through T confocal with the given one.

Hint. Use the result of Example 4 of § 15.4

6. Show that the locus of the points of contact of tangents drawn through a fixed point (α, β) to the system of conics confocal with $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is the cubic curve.

$$\frac{x}{y-\beta}+\frac{y}{x-\alpha}=\frac{a^2-b^2}{\alpha y-\beta x},$$

which passes through the foci of the confocals. (*Lucknow*, 1979)

7. Show that points of contact of the tangents of the confocal conics

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

which are also tangents to the parabola

$$y^2=4x\sqrt{c^2-b^2},$$

lie on a straight line. (*Lucknow*, 1980)

8. TP, TP' are tangents one to each of two confocal conics whose centre is C; if the angle PTP' be a right angle, show that CT will bisect PP'.

9. Show that the equation of the pair of tangents from P to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

referred to the normals to the confocals through P as axes is

$$\frac{X^2}{\lambda}+\frac{Y^2}{\mu}=0,$$

where λ, μ are the parameters of the confocals through P.

10. Show that the ends of the equal conjugate diameters of a series of confocal ellipse are on a confocal rectangular hyperbola.

11. If the product of the perpendiculars let fall on a straight line from its pole with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and from the centre of the ellipse is a constant quantity λ, then prove that the straight line is a tangent to the confocal

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

12. An ellipse and a hyperbola are confocal and the asymptotes of the hyperbola lie along the equi-conjugate diameters of the ellipse. Prove that the hyperbola will cut orthogonally all conics which pass through the ends of the axes of the ellipse.

13. Show that only one of a given system of confocals can have a given straight line as a normal.

14. If two confocal conics intersect, prove that the centre of curvature of either curve at a point of intersection is the pole of the tangent at the point with respect to the other curve.

15. Prove that the locus of the pole of the axis of x with respect to the circle of curvature at any point of the parabola $y^2=4ax$ is

$$(x-2a)^3 y^2=12a(x^2-ax+a^2)^2.$$

16. The centres of curvature at the points 't_1', 't_2', 't_3' on the parabola $y^2=4ax$ are collinear, prove that

$$\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}=0.$$

17. The circle of curvature of the rectangular hyperbola $x^2-y^2=a^2$ at the point (a cosec α, a cot α) meets the curve again in the point (a cosec β, a cot β). Show that

$$\tan^3\frac{\alpha}{2}\tan\frac{\beta}{2}-1=0.$$

18. Prove that the radical axis of the circles of curvature drawn to a hyperbola and its conjugate at the ends of conjugate diameters is parallel to one of the asymptotes.

19. A parabola has double contact with a given ellipse. If its axis be parallel to a given straight line, prove that the locus of its focus is a hyperbola confocal with the ellipse and having one asymptote in the given direction.

20. Find the locus of points whose polar lines with respect to the conics

$$\frac{x^2}{\alpha}+\frac{y^2}{\beta}=1, \quad (\alpha+\beta\neq 0) \qquad \text{...(1)}$$

and

$$ax^2+hxy+by^2=1 \qquad \text{...(2)}$$

are perpendiculars.

Hence, or otherwise, show that the conics having centre at the origin which meet (1) orthogonally at four points are either confocal with it or else pass through four fixed points on its principal axes.

Ans. $a\beta x^2+(\alpha+\beta)hxy+bxy^2=0.$

21. Normals are drawn from a point P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If λ, μ be the parameters of the confocals to the ellipse through P, show that the product of the normals is

$$\frac{\lambda\mu(\lambda-\mu)}{a^2-b^2}.$$

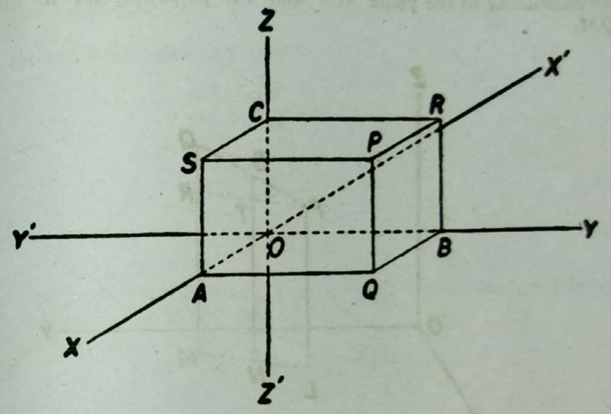
Hint. Four normals can be drawn from a point to an ellipse.

Three Dimensions

CHAPTER XVI

PLANE, STRAIGHT LINE, SPHERE

16.1 Coordinates of a point in space. To fix the position of a point in space we require three concurrent lines which are not coplanar. Let $X'OX$, $Y'OY$, $Z'OZ$ be such straight lines whose positive directions are $X'OX$, $Y'OY$, $Z'OZ$.



Let P be a point in space and let planes parallel to the planes YOZ, ZOX, XOY be drawn through P to meet lines $X'X$, $Y'Y$, $Z'Z$ in A, B, C, then the position of P is known when the segments OA, OB, OC are given in magnitude and sign. If $OA=x$, $OB=y$, $OC=z$, we say that (x, y, z) are then **cartesian coordinates** of P. The lines $X'OX$, $Y'OY$, $Z'OZ$ are called the coordinate axes and the planes YOZ, ZOX, XOY, the coordinate planes. The point O is called the **origin**.

The coordinate planes divide space into eight parts called **octants**, and the signs of the coordinates of a point determine the octant in which it lies.

It is convenient to choose multually perpendicular lines as coordinate axis. The axes are then said to be **rectangular.**

Note. The positive directions of a system of rectangular axes are assigned in accordance with the usual convention made in plane

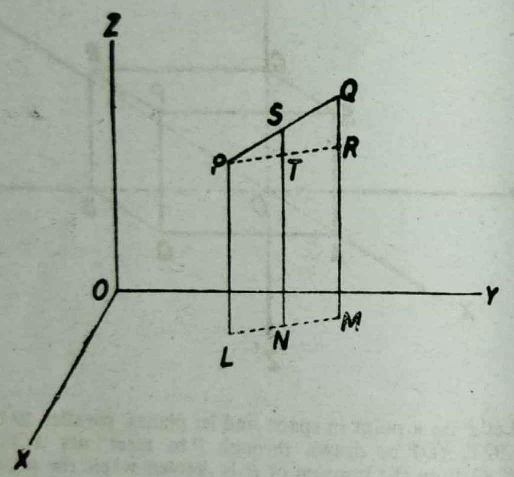
geometry. Thus, the positive direction of rotation for a ray in any plane is fixed accordance to following rule :

If a clock dial is considered to coincide with the plane and front in the positive direction of the normal to the plane, the positive direction of rotation for a ray revolving in the plane is counter-clock wise.

Applying this rule to the coordinate planes, we see that the positive directions of rotation for the planes XOY, YOZ, ZOX are $XYX'Y'$, $YZY'Z'$, $ZXZ'X'$.

16.12 Distance between two given points.

Let P, Q be the points (x_1, y_1, z_1), (x_2, y_2, z_2). Draw PL, QM perpendicular to the plane XOY, and PR perpendicular to the line QM.



Then, obviously, the coordinates of L, M in the xy-plane are (x_1, y_1), (x_2, y_2), and

$$LM^2=(x_2-x_1)^2+(y_2-y_1)^2.$$

Further $\quad QR=QM-RM=QM-PL=z_2-z_1.$

Now, $\quad PQ^2=PR^2+QR^2=LM^2+QR^2$

$$=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2.$$

Corollary. The distance of a given point (x, y, z) from the origin is

$$\sqrt{x^2+y^2+z^2}.$$

16.13 Coordinates of the point which divides in a given ratio the line joining two given points.

Referred to the figure of the preceding article, let S divide PQ in the ratio $\lambda : 1$. Draw SN perpendicular to the xy-plane meeting PR in T. Then N divides LM in the ratio $\lambda : 1$, and the coordinates of N, in the xy-plane, are

$$\left(\frac{x_1+\lambda x_2}{1+\lambda},\ \frac{y_1+\lambda y_2}{1+\lambda}\right)$$

If (x, y, z) *are the coordinates of* S, $x=\dfrac{x_1+\lambda x_2}{1+\lambda}$; $y=\dfrac{y_1+\lambda y_2}{1+\lambda}$,

The z-coordinate of T is z_1, and therefore $ST=z-z_1$.

From the triangles PST, PQR,

$$\frac{ST}{QR}=\frac{PS}{PQ}.$$

i.e.,
$$\frac{z-z_1}{z_2-z_1}=\frac{\lambda}{1+\lambda},$$

or
$$z=\frac{z_1+\lambda z_2}{1+\lambda}.$$

Hence, **the coordinates of a point which divides the line joining (x_1, y_1, z_1), (x_2, y_2, z_2) in the ratio $\lambda : 1$ are**

$$\left(\frac{x_1+\lambda x_2}{1+\lambda},\ \frac{y_1+\lambda y_2}{1+\lambda},\ \frac{z_1+\lambda z_2}{1+\lambda}\right)$$

Note. If λ is negative then S divides PQ externally in the ratio $\lambda : 1$.

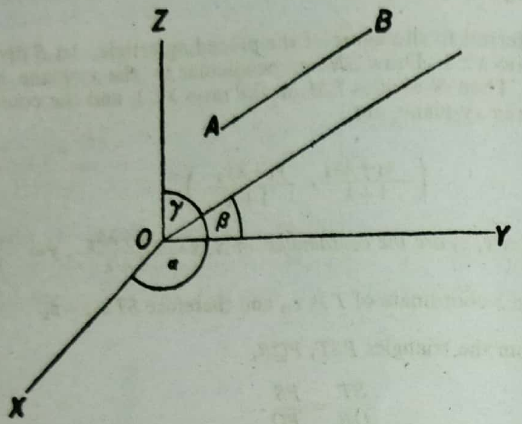
Corollary. The coordinate of the middle point of the line joining the points (x_1, y_1, z_1), (x_2, y_2, z_2) are

$$\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2},\ \frac{z_1+z_2}{2}\right).$$

16.2 Directions cosines of a line.

Let AB be a given straight line. Through O draw a line parallel to AB. The angle which AB makes with the coordinate axes are the same as those made by the parallel straight line. Denoting these angles by α, β, γ, we say that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of AB.

The direction cosines of a line are usually denoted by the letters l, m, n.



Corollary. The direction cosines of the axes of x, y, z are respectively 1, 0, 0; 0, 0, 1.

Note. Quantities proportional to direction cosines are called **direction ratios.**

16.21 Relation between the direction cosines of a given line.

We shall prove that if l, m, n *be the direction cosines of a given line then* $l^2+m^2+n^2=1$.

Let l, m, n be the direction cosines of a given line. The direction cosines of OP which is drawn parallel to the given line are then also l, m, n.

Draw PA perpendicular to OX. If (x, y, z) be the coordinates of P, then $OA=x$.

Let $OP=r$, and let the angle POA be α, Then, from the right angled triangle AOP.

$$\frac{AO}{OP}=\cos\alpha.$$

i.e.,
$$\frac{x}{r}=1,$$

or
$$x=lr.$$

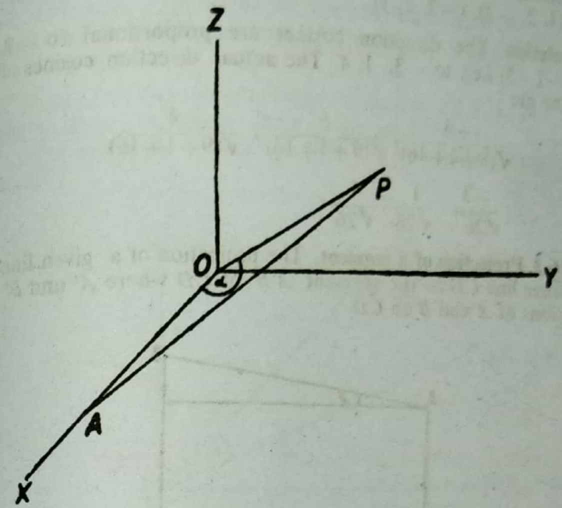
Similarly
$$y=mr,\ z=nr.$$

From these, on squaring and adding,

$$x^2+y^2+z^2=r^2(l^2+m^2+n^2).$$

But
$$x^2+y^2+z^2=r^2.$$

Therefore,
$$l^2+m^2+n^2=1.$$



16.22 Direction cosines of the line joining two given points (x_1, y_1, z_1), (x_2, y_2, z_2).

Let P, Q be the points (x_1, y_1, z_1), (x_2, y_2, z_2), and let PQ be equal to r.

Then
$$r^2=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2.$$

Transferring the origin to P, the axes remaining parallel to original axes, the coordinates of Q are $(x_2-x_1,\ y_2-y_1,\ z_2-z_1)$. If l, m, n be the direction cosines of PQ, we have from the preceding article

$$\frac{x_2-x_1}{r}=1,$$

$$\frac{y_2-y_1}{r}=m,$$

and
$$\frac{z_2-z_1}{r}=n.$$

The direction cosines of the given line are thus proportional to the quantities $x_2-x_1, y_2-y_1, z_2-z_1$, their actual values being

$$\frac{x_2-x_1}{r},\ \frac{y_2-y_1}{r},\ \frac{z_2-z_1}{r}$$

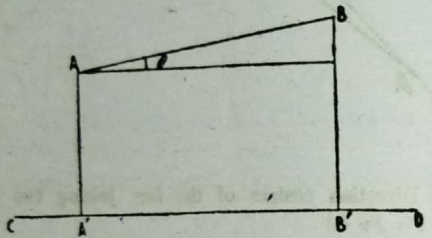
Example. Find the direction cosines of the line joining the points (1, 2, —3), (—2, 3, 1).

Solution. The direction cosines are proportional to $-2-1$, $3-2$, $1-(-3)$ i.e., to -3, 1, 4. The actual direction cosines of the given line are

$$\frac{-3}{\sqrt{(9+1+16)}}, \frac{1}{\sqrt{(9+1+16)}}, \frac{4}{\sqrt{(9+1+16)}}$$

i.e.,
$$\frac{-3}{\sqrt{26}}, \frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}.$$

16.3 Projection of a segment. The projection of a given line AB on another line CD is the segment $A'B'$ of CD where A' and B' are projections of A and B on CD.



To find the projection of AB on CD, draw planes through A and B each perpendicular to CD intersecting CD in A' and B'. If θ is the angle between AB and CD, the length $A'B'$ of the projection is obviously $AB \cos \theta$.

16.31 A theorem on projection. We shall now prove the following theorem : from notes.

If A, B, C, ...M, N are any n points in the space, the sum of the projections of AB, BC, ...MN on any given line PQ is equal to the projection of the straight line AN on PQ.

Let $A', B', C' \ldots M', N'$ be the feet of the perpendiculars from $A, B, C, \ldots M, N$ on PQ.

Then the projections of $AB, BC, \ldots MN$ on PQ are respectively $A'B', B'C', \ldots M'N'$, and the projection of AN on PQ is $A'N'$.

But
$$A'B'+B'C'+\ldots+M'N'=A'N'.$$

This establishes theorem.

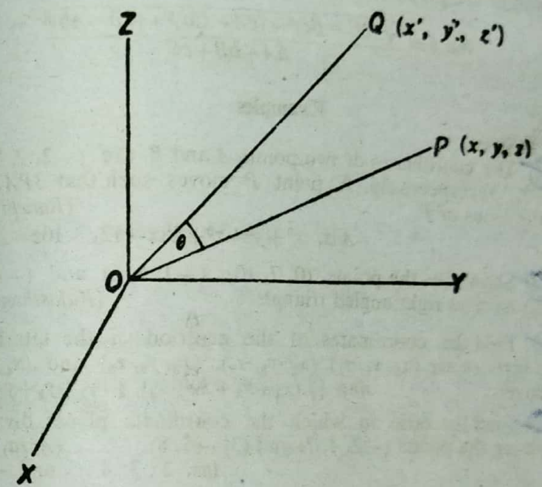
Note. In determining the projection of the one line on another care must be taken regarding the sense of rotation.

For an acute angle the projection is positive or negative according as the rotation is counter-clockwise or clockwise.

16.4 Angles between two lines.

Let l, m, n; l', m', n' be the direction cosines of two given lines AB, CD. We shall find the angle between AB and CD in terms of their direction cosines.

Draw OP, OQ parallel respectively to AB, CD. The angle θ between OP and OQ is the same as the angle between AB and CD. Also, the direction cosines of OP, OQ are respectively l, m, n; l', m', n'.



Let the coordinates of P and Q be respectively (x, y, z) and (x', y', z'). If $OQ=r'$, the projection of OQ on OP is $r' \cos \theta$. From § 16. 31, this projection is also equal to $lx'+my'+nz'$.

Therefore,

$$r' \cos \theta=lx'+my'+nz',$$

or
$$\cos \theta=l \frac{x'}{r'}+m \frac{y'}{r'}+n \frac{z'}{r'}$$
$$=ll'+mm'+nn'.$$

The angle θ between the lines is given by

$$\cos \theta=\mathbf{ll'+mm'+nn'}.$$

Corollary 1. From Lagrange's identity

$$(l^2+m^2+n^2)(l'^2+m'^2+n'^2)-(ll'+mm'+nn')^2$$
$$=(mn'-nm')^2+(nl'-ln')^2+(lm'-ml')^2.$$

Scanned by CamScanner

$$\sin^2\theta=1-\cos^2\theta$$
$$=(l^2+m^2+n^2)\,(l'^2+m'^2+n'^2)-(ll'+mm'+nn')^2$$
$$=(mn'-nm')^2+(nl'-ln')^2+(lm'-ml')^2.$$

Corollary 2. The lines are perpendicular if $ll'+mm'+nn'=0$, and parallel if

$$\frac{l}{l'}=\frac{m}{m'}=\frac{n}{n'}.$$

Corollary 3. If θ be the angle between the lines whose direction cosines are proportional to a, b, c and A, B, C, then

$$\tan\theta=\frac{\sqrt{(bC-Bc)^2+(cA-Ca)^2+(aB-Ab)^2}}{aA+bB+cC}$$

Examples

1. The coordinates of two points A and B are $(-2, 2, 3)$ and $(13, -3, 13)$ respectively. A point P moves such that $3PA=2PB$. Find the locus of P. *(Ranchi, 1970)*

Ans. $x^2+y^2+z^2+28x-12y+10z-247=0$.

2. Show that the points $(0, 7, 10)$, $(-1, 6, 6)$ and $(-4, 9, 6)$ form an isoceles right angled triangle. *(Rajasthan, 1973)*

3. Find the coordinates of the centriod of the tetarhedron whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) respectively. *Ans.* $[\frac{1}{4}(x_1+x_2+x_3+x_4), \frac{1}{4}(y_1+y_2+y_3+y_4)]$.

imp 4. Find the ratio in which the coordinate planes divide the line joining the points $(-2, 4, 7)$ and $(3, -5, 8)$. *(Agra, 1977)*

Ans. $2:3$; $4:5$ and $-7:8$.

5. Find the coordinates of the circumcentre of the triangle formed by the points whose vertices are $(1, 1, 0)$, $(1, 2, 1)$ and $(-2, 2, -1)$. *(Allahabad, 1966)*

Ans. $(-\frac{1}{2}, 2, 0)$.

6. If α, β, γ be the angles which a line makes with the coordinate axes, show that

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2.$$

$1-\cos^2\alpha.$

7. Show that the direction cosines of the line equally inclined to the coordinate axes are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$. $l^2+m^2+n^2=1$

8. Find the angle between the lines whose direction cosines are proportional to 1, 2, 4 and -2, 1, 5.

Ans. $\cos^{-1}\left(\frac{20}{3\sqrt{70}}\right)$.

9. Find the direction cosines of the line which is perpendicular to the line with direction cosines proportional to $3, -1, 1$ and $-3, 2, 4$.

Ans. $\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{33}}, \frac{-1}{\sqrt{30}}$.

Hint. If the direction cosines of the required line are proportional to l, m, n, then

$$3l-m+n=0 \text{ and } -3l+2m+4n=0.$$

10. Three concurrent lines with direction cosines proportional to l_r, m_r, n_r $(r=1, 2, 3)$ are coplanar. Show that

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=0.$$

(Meerut, 1975; Agra, 1977)

Hint. The lines are coplanar if there is a line perpendicular to each of them.

11. Show that the line joining the points $(1, 2, 3)$, $(-1, -2, -3)$ is parallel to the line joining the points $(2, 3, 4)$, $(5, 9, 13)$ and perpendicular to the line joining $(-2, 1, 5)$ and $(3, 3, 2)$.

12. Find the angle between the lines whose direction cosines are given by the equation

$$3l+m+5n=0 \text{ and } 6mn-2nl+5lm=0.$$

(Lucknow, 1979)

Ans. $\cos^{-1}(\frac{1}{6})$.

13. Find the angle between the lines whose direction cosines satisfy the equation $l+m+n=0$, $2lm-mn+2nl=0$. *(Agra, 1971)*

Ans. $60°$.

14. Show that the acute angle between the diagonals of a cube is $\cos^{-1}(\frac{1}{3})$. *(Ranchi, 1970)*

15. A line makes angles $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube. Prove that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\tfrac{4}{3}.$$ *(Vikram, 1971; Kanpur, 1976)*

Also show that

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma+\sin^2\delta=\tfrac{8}{3}.$$

(Gorakhpur, 1970; Rajasthan, 1976; Lucknow, 1980)

16. If A, B, C, D are the points $(3, 4, 5)$, $(4, 6, 3)$, $(-1, 2, 4)$ and $(1, 0, 5)$, find the projection of CD on AB.

(*Rohilkhand*, 1977)

Ans. (4/3) units.

17. Lines OP and OQ are drawn from O with direction cosines proportional to $1, -2, 1$; $7, -6, 1$. Find the direction cosines of the normal to the plane OPQ.

Ans. $\dfrac{2}{\sqrt{29}}, \dfrac{3}{\sqrt{29}}, \dfrac{4}{\sqrt{29}}$

18. If the edges of a rectangular parallelopiped are a, b, c; show that the angles between the four diagonals are given by

$$\cos^{-1}\left[\frac{a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right].$$ (*Kanpur*, 1977)

19. If l_r, m_r, n_r $(r=1, 2, 3)$ be the direction cosines of three mutually perpendicular lines, then prove that the line whose direction cosines are proportional to $l_1+l_2+l_3$, $m_1+m_2+m_3$, $n_1+n_2+n_3$ makes equal angles with them. (*Rajasthan*, 1971)

16.5 Distance of a point from a line.

Let P be the point (x', y', z') and let the direction cosines of a line drawn through another point A, (α, β, γ) be l, m, n. Let the length of the perpendicular PN from P upon the line be p.

Now AN is the projection of AP on the line. Therefore,

$$AN=(x'-\alpha)\, l+(y'-\beta)\, m+(z'-\gamma)\, n.$$

But $\quad PN^2=AP^2-AN^2.$

Therefore, $p^2=(x'-\alpha)^2+(y'-\beta)^2+(z'-\gamma)^2$

$$-\{(x'-\alpha)\, l+(y'-\beta)\, m+(z'-\gamma)\, n\}^2,$$

or, using Lagrange's identity,

$$p^2=\{(y'-\beta)\, n-(z'-\gamma)m\}^2+\{(z'-\gamma)\, l-(x'-\alpha)\, n\}^2$$
$$+\{(x'-\alpha)\, m-(y'-\beta)\, l\}^2.$$

Ex. 1. Find the distance of the point $(1, 2, 3)$ from the line through the points $(-1, 2, 5)$, and $(2, 3, 4)$.

Ans. $2\sqrt{6}/\sqrt{11}$.

Hint. The direction cosines of the line are proportional to $3, 1, -1$.

Ex. 2. Show that the distance of the point $(1, -2, 3)$ from the line which is equally inclined to the axes and which passes through the point $(2, -3, 5)$ is $\sqrt{14/3}$.

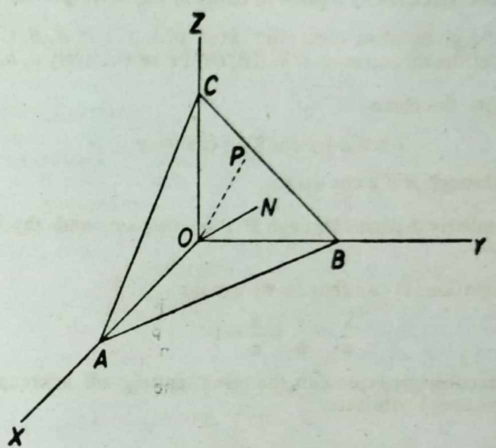
16.6 Equation of a plane. Let the perpendicular ON from the origin upon a given plane have direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and let its measure be p.

Let P be any point (x, y, z) on the plane.

The projection of OP on ON is ON itself, i.e., p.

This projection is also equal to $x \cos\alpha+y\cos\beta+z\cos\gamma$. Hence the equation to the plane is

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p.$$



Now the general equation of the first degree

$$Ax+By+Cz+D=0$$

can be written as

$$\frac{A}{\sqrt{(A^2+B^2+C^2)}}\,x+\frac{B}{\sqrt{(A^2+B^2+C^2)}}\,y+\frac{C}{\sqrt{(A^2+B^2+C^2)}}\,z$$
$$=-\frac{D}{\sqrt{(A^2+B^2+C^2)}},$$

i.e., as

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p \qquad \text{...(1)}$$

where $\cos\alpha=\dfrac{A}{\sqrt{(A^2+B^2+C^2)}}$, $\cos\beta=\dfrac{B}{\sqrt{(A^2+B^2+C^2)}}$,

$\cos\gamma=\dfrac{C}{\sqrt{(A^2+B^2+C^2)}}$, and $p=-\dfrac{D}{\sqrt{(A^2+B^2+C^2)}}$,

it being supposed that D is negative (p is a positive number). If D is positive, we mus change the sign of $\sqrt{(A^2+B^2+C^2)}$.

Equation (1) represents a plane. *We thus conclude that the general equation of the first degree in x, y, z represents a plane.*

Corollary. The direction cosines of the normal to the plane $Ax+By+Cz+D=0$ are proportional to A, B, C.

16.61 Equation of a plane in terms of the intercepts on the axes.

Let a given plane meet the axes of x, y, z in A, B, C respectively. Let the measures of OA, OB, OC be respectively a, b, c.

Now, the plane

$$x \cos \alpha+y \cos \beta+z \cos \gamma=p \qquad \ldots(1)$$

passes through A if $a \cos \alpha=p$.

Similarly it passes through B if $b \cos \beta=p$, and through C if $c \cos \gamma=p$.

Equation (1) can thus be written as

$$\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}+\frac{\mathbf{z}}{\mathbf{c}}=\mathbf{1}$$

which accordingly represents the plane cutting off intercept a, b, c from the coordinate axes.

Examples

1. Find the distance of the origin from the plane

$$2x+6y-3z+7=0. \qquad \textit{Ans. } 1.$$

2. Show that the plane whose equations are $2x+3y-z+1=0$ and $x-2y-4z+5=0$ are mutually perpendicular.

3. Find the angle between the planes $x+y+z+1=0$ and $2x-y+4z+4=0$.

$x-y+4z+4=0$

$\cos\theta = \left|\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right|$ *Ans* $\cos^{-1}\dfrac{5}{3\sqrt{7}}$.

4. Find the equation to the plane through $(1, 2, -1)$ parallel to $x+2y+3z+4=0$.

Ans. $x+2y+3z=2$.

5. O is the origin and P the point $(-1, 1, -3)$. Find the equation to the plane through P at right angles to OP.

Ans. $x-y+3z+11=0$.

6. A plane meets the coordinate axes in A, B, C, such that the centroid of the triangle ABC is the point (p, q, r). Show that the equation to the plane is

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3.$$

(*Allahabad*, 1965; *Kanpur*, 1976)

7. Find the equation to the plane through the points $(-1, 1, 1)$ and $(1, -1, 1)$ and perpendicular to the plane $x+2y+2z=5$.

(*Gorakhpur*, 1970; *Rajasthan*, 1971)

Ans. $2x+2y+3z+3=0$.

8. Find the locus of a point such that the sum of the squares of its distances from the planes $x+y+z=0$ and $2x-2y+z=0$ is equal to its distance from the plane $x=z$. (*Agra*, 1971)

Ans. $y^2+zx=0$.

9. Find the equation of the plane which passes through the point $(-1, 3, 2)$ and is perpendicular to each of the two planes

$$x+2y+2z=5 \text{ and } 3x+3y+2z=8.$$

(*Delhi*, 1971; *Gorakhpur*, 1973)

10. A variable plane is at a constant distance p from the origin and meets the coordinate axes in A, B and C. Through A, B, C, planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$$

(*Lucknow*, 1970; *Meerut*, 1974; *Kanpur*, 1975)

11. Find the equation of the plane passing through the line of intersection of the planes $ax+by+cz+d=3$ and $a'x+b'y+c'z+d'=0$ and perpendicular to xy-plane. (*Kurukshetra*, 1973)

Ans. $(ac'-a'c)\,x+(bc'-b'c)\,y+(dc'-d'c)=0$.

12. Find the equation to the plane passing through the points $(2, 3, -4)$ and $(1, -1, 3)$ and parallel to x-axis

(*Gorakhpur*, 1976)

Ans. $7y+4z-5=0$.

16.62 Plane through three given points. We shall now find the equation to the plane which passes through three non-collinear points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3).

Let the equation to the plane be

$$ax+by+cz+d=0. \qquad \ldots(1)$$

Now equation (1) is satisfied by the coordinates of the given points. Therefore,

$$ax_1+by_1+cz_1+d=0, \quad \ldots(2)$$
$$ax_2+by_2+cz_2+d=0, \quad \ldots(3)$$
and
$$ax_3+by_3+cz_3+d=0. \quad \ldots(4)$$

Eliminating a, b, c, d from (1), (2), (3) and (4), the required equation to the plane is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0.$$

16.63 Length of the perpendicular from a point to a plane.

We shall now find the distance of the point $P:(x', y', z')$ from the plane

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction consines of the perpendicular from the origin upon the plane.

Let us transfer the origin to (x', y', z'). The equation to the plane now becomes

$$(x+x')\cos\alpha+(y+y')\cos\beta+(z+z')\cos\gamma=p,$$

i.e.,

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p-x'\cos\alpha-y'\cos\beta-z'\cos\gamma.$$

Hence the distance of (x', y', z'), the new origin, from the plane is

$$p-x'\cos\alpha-y'\cos\beta-z'\cos\gamma.$$

Note. The length of perpendicular from a point to a given plane is always written with plus sign even if the corresponding expression has a negative value.

Corollary. The length of the perpendicular from (x', y', z') to the plane $ax+by+cz+d=0$ is the numerical value of the expression

$$\frac{ax'+by'+cz'+d}{\sqrt{(a^2+b^2+c^2)}}.$$

16.64 Planes bisecting the angles between two given planes.

Let the equations to the given planes be

$$ax+by+cz+d=0$$
and
$$a'x+b'y+c'z+d'=0.$$

If (x, y, z) be any point on the plane bisecting the angle between the given planes, then the lengths of the perpendiculars dropped from this point on the two planes will be equal in magnitude. Hence,

$$\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}}=\pm\frac{a'x+b'y+c'z+d'}{\sqrt{a'^2+b'^2+c'^2}}$$

are the equations of the bisecting planes, one bisecting the acute angle and the other the obtuse angle between the given planes.

Note. To determine which of the two bisecting planes bisects the angle containing the origin, we proceed as in two dimensions.

The student should find no difficulty in working out the details.

Examples

1. Show that the plane $ax+by+cz+d=0$ divides the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio

$$-(ax_1+by_1+cz_1+d):(ax_2+by_2+cz_2+d).$$

2. Find the equation of the plane passing through the line of intersecting of the planes $x+y+z=6$ and $2x+3y+4z+5=0$ and peryendicular to the plane $4x+5y-3z=8$. (*Lucknow*, 1969)

Solution. The equation of the plane passing through the given line is

$$(2x+3y+4z+5)+\lambda(x+y+z-6)=0$$

or

$$x(2+\lambda)+y(3+\lambda)+z(4+\lambda)+5-6\lambda=0. \quad \ldots(1)$$

This is perpendicular to the third plane. Therefore

$$4(2+\lambda)+5(3+\lambda)-3(4+\lambda)=0$$

giving

$$\lambda=-\frac{11}{6}.$$

Putting this value of λ in (1), the required equation is

$$x+7y+13z+96=0.$$

3. Prove that the planes

$$7x+4y-4z+30=0,$$
$$36x-51y+12z+17=0,$$
$$14x+8y-8z-12=0,$$
and
$$12x-17y+4z-3=0,$$

from the four faces of a rectangular parallelopipe.

4. Find the equation of the plane which bisects the obtuse angle between the planes $3x+4y-5z+1=0$ and $5x+12y+-13z=0$

(*Agra*, 1971)

Ans. $14x-8y+13=0$

5. Show that the origin lies in the acute angle between the planes $x+2y+2z-9=0$ and $4x-3y+12z+13=0$. Find the planes bisecting the angles between them and the one which bisects the acute angle.

(*Ravishankar*, 1970; *Rajasthan*, 1973)

Ans. Bisector of the acute angle : $25x+17y+62z-78=0$.

Other plane : $x+35y-10z-156=0$.

7. Prove that the equation

$$ax^2+by^2+cz^2+2fyz+2gxz+2hxy=0$$

represents a pair of planes passing through the origin, if

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

Prove also that the angle between the planes is

$$\tan^{-1}\left[\frac{2\{f^2+g^2+h^2-bc-ca-ab\}^{1/2}}{a+b+c}\right]$$

(*Agra*, 1973; *Kanpur*, 1977; *I. A. S.*, 1976)

8. Prove that the equation

$$2x^2+6y^2-12z^2+yz+2zx+7xy=0$$

represents a pair of planes passing through the origin and find the angle between them. (*Gorakhpur*, 1970; *Vikram*, 1971)

Ans. $\tan^{-1}\left(\frac{\sqrt{390}}{4}\right)$

9. Prove that the equation

$$\frac{a}{y-z}+\frac{b}{z-x}+\frac{c}{x-y}=0$$

represents a pair of planes.

(*Agra*, 1974; *Rajasthan*, 1976; *U. P. C. S.*, 1976)

10. Prove that $\frac{3}{y-z}+\frac{4}{z-x}+\frac{5}{x-y}=0$ represents a pair of planes.

(*Kurukshetra*, 1974)

11. A plane makes intercepts $OA=a$, $OB=b$, $OC=c$ respectively on the coordinate axes, where O is the origin. Find the area of the triangle ABC.

(*Kanpur*, 1977)

A_{n}. $\frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$

12. Through a point $P:(\alpha, \beta, \gamma)$, a plane is drawn at right angles to OP to meet the coordinate axes in A, B and C respectively. If $OP=p$, show that the area of the triangle ABC is

$$\frac{p^5}{2\alpha\beta\gamma}.$$

(*Rajasthan*, 1970)

13. Two systems of rectangular axes have the same origin. If a plane cuts them at distance a, b, c and a', b', c' from the origin, show that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}.$$

(*Kanpur*, 1976; *Lucknow*, 1978)

Solution. Let the equation of the plane referred to the sets of rectangular axes be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \text{ and } \frac{X}{a'}+\frac{Y}{b'}+\frac{Z}{c'}=1.$$

The length of the perpendicular from the origin will be the same for the two sets of axes. Therefore,

$$\frac{1}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}=\frac{1}{\sqrt{\left(\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}\right)}}$$

The result now follows immediately.

14. A variable plane which remains at a constant distance $3p$ from the origin, cuts the coordinate axes in A, B, C. Show that the locus of the centroid of the triangle ABC is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$$

(*Kurukshetra*, 1976)

15. A variable plane is at a constant distance p from the origin and meets the coordinate axes in A, B, C. Show that the locus of the centroid of the tetrahedron $OABC$ is

$$x^{-2}+y^{-2}+z^{-2}=16p^{-2}.$$

(*Kanpur*, 1970; *Rajasthan*, 1974)

16. A point P moves on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, which is fixed and the plane through P perpendicular to OP meets the coordinates axes in A, B, C. If the planes through A, B, C parallel to coordinate planes meet in some point Q, show that the locus of Q is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}.$$

(*Rajasthan*, 1968)

Hint. Let the coordinates of the point P be (α, β, γ). Then

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1. \qquad \ldots(1)$$

If (x_1, y_1, z_1) be the coordinates of the point Q, then

$$x_1=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha},\ y_1=\frac{\alpha^2+\beta^2+\gamma^2}{\beta},\ z_1=\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}.$$

Eliminating α, β, γ, between these relations and (1).

17. A triangle, the length of whose sides are are a, b, c, is placed so that the middle points of the sides are on the coordinate axes. Show that the equation to the plane is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1,$$

where $8\alpha^2=b^2+c^2-a^2$, $8\beta^2=c^2+a^2-b^2$, and $8\gamma^2=a^2+b^2-c^2$.

Also find the coordinates of the vertices.

(*Allahabad*, 1968; *Rajasthan*, 1970; *U. P. C. S.*, 1971)

Solution. Let the equation to the required plane be

$$\frac{x}{A}+\frac{y}{B}+\frac{z}{C}=1. \qquad \ldots(1)$$

This meets the coordinate axes in $(A, 0, 0)$, $(0, B, 0)$ and $(0, 0, C)$. Since the line joining the middle points of two sides of a triangle is parallel to third side and equal to half of it, we have

$$B^2+C^2=\frac{1}{4}a^2,\ C^2+A^2=\frac{1}{4}b^2,\ A^2+B^2=\frac{1}{4}c^2.$$

These give $A^2=\frac{1}{8}(b^2+c^2-a^2)=\alpha^2$, etc.

Hence the equation of the plane is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1.$$

Further, let the coordinates of A be (x_1, y_1, z_1). The coordinates of B and C will be $(-x_1, -y_1, 2\gamma-z_1)$ and $(-x_1, 2\beta-y_1, -z_1)$ because of the middle point of AB is $(0, 0, \gamma)$ and that of AC is $(0, \beta, 0)$.

The middle point of BC will be $(-x_1, \beta-y_1, \gamma-z_1)$ which is the same as $(\alpha, 0, 0)$. Hence $x_1=-\alpha$, $y_1=\beta$, $z_1=\gamma$. Similarly, the coordinates of B and C are $(\alpha, -\beta, \gamma)$ and $(\alpha, \beta, -\gamma)$.

16.7 The straight line. We know that an equation of the first degree will always represents a plane. Consider now the two equations

$$ax+by+cz+d=0. \qquad a'x+b'y+c'z+d'=0.$$

They are satisfied by the coordinates of any point on the line of intersection of the planes which they represent.

We thus see that two equations of the first degree together represent a straight line.

The equations to a straight line can be obtained in a more symmetrical form.

If P is a given point (α, β, γ) on a line which has direction cosines l, m, n, then from § 16.22, we have

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r, \qquad \ldots(1)$$

where (x, y, z) is any point Q on the line, and r is the measure of PQ.

The equations to a line passing through a given point (α, β, γ) is, therefore,

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

where l, m, n may now be quantities proportional to the direction cosines of the line.

If l, m, n are the actual direction cosines of the line, the coordinates of a variable point on the line distance r from the fixed point (α, β, γ) are

$$x=\alpha+lr,\ y=\beta+mr,\ z=\gamma+nr.$$

Corollary 1. The equations of a line joining the fixed points (α, β, γ), $(\alpha', \beta', \gamma')$ are

$$\frac{x-\alpha}{\alpha'-\alpha}=\frac{y-\beta}{\beta'-\beta}=\frac{z-\gamma}{\gamma'-\gamma},$$

and the coordinates of a variable point on the line, in terms of a single parameter λ, are

$$x=\frac{\lambda\alpha'+\alpha}{\lambda+1},\ y=\frac{\lambda\beta'+\beta}{\lambda+1},\ z=\frac{\lambda\gamma'+\gamma}{\lambda+1}.$$

Corollary 2. The equations of the coordinate axes are $y=0, z=0$; $z=0, x=0$; $x=0, y=0$.

Examples

1. Find the coordinates of the point in which the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z+2}{3}$ meets the plane $2x+3y+z=0$.

Ans. $\left(-\frac{1}{11}, \frac{9}{11}, -\frac{25}{11}\right)$

2. Find the distance from the point $\left(2, -\frac{3}{2}, -1\right)$ to the point where the line $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z+3}{4}$ meets the plane $2x+4y+z+1=0$.

Ans. $\frac{\sqrt{205}}{2}$.

3. Show that the equations to the line through (α, β, γ) at right angles to the lines

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}, \quad \frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}$$

are $$\frac{x-\alpha}{mn'-nm'}=\frac{\gamma-\beta}{nl'-ln'}=\frac{z-\gamma}{lm'-ml'}.$$

Hint. If λ, μ, ν be the direction ratios of the required line, $l\lambda+m\mu+n\nu=0,\ l'\gamma+m'\mu+n'\nu=0$.

4. Prove that the equation to the line of intersection of the planes $4x+4y-5z=12$, $8x+12y-13z=32$ can be written as

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}.$$ (*Lucknow*, 1979)

16.71 Straight line lying in a plane. Let

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

be the equation of a straight line, and

$$ax+by+cz+d=0$$

the equation of a plane.

The point of intersection of the straight line and the plane is

$$(\alpha+lr, \beta+mr, \gamma+nr),$$

where r is given by

$$r\,(al+bm+cn)+a\alpha+b\beta+c\gamma+d=0.$$

Now, r is proportional to the distance of the point from (α, β, γ). The line therefore lies in the plane if

$$al+bm+cn=0,$$

and $$a\alpha+b\beta+c\gamma+d=0.$$

Corollary. The conditions that the line should be parallel to the plane, are $al+bm+cn=0$ and $a\alpha+b\beta+c\gamma+d\neq 0$.

16.72 Coplanar lines. Let

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad ...(1)$$

and $$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'} \quad ...(2)$$

be two straight lines. We shall find the condition that these straight lines be coplanar.

The equation to a plane containing the first line is

$$a\,(x-\alpha)+b\,(y-\beta)+c\,(z-\gamma)=0 \quad ...(3)$$

where $$al+bm+cn=0. \quad ...(4)$$

If the plane contains the second line.

$$a\,(\alpha'-\alpha)+b\,(\beta'-\beta)+c\,(\gamma'-\gamma)=0 \quad ...(5)$$

and $$al'+bm'+cn'=0. \quad ...(6)$$

Eliminating a, b, c between (4), (5) and (6), the condition that the given lines be coplanar is

$$\begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0.$$

On eliminating a, b, c between (3), (4) and (6), we get the equation to the plane containing the lines as

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0.$$

Examples

1. Prove that the line $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}$ lies in the plane $4x+4y+5z-3=0$. (*Lucknow*, 1978)

2. Find the equation of the plane which passes through the line $ax+by+cz+d=0,\ a'x+b'y+c'z+d'=0$ and is parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$ (*Lucknow*, 1953)

Solution. The equation of a plane passing through the given line is

$$(ax+by+cz+d)+\lambda\,(a'x+b'y+c'z+d')=0,$$

i.e.,
$$(a+\lambda a')x+(b+\lambda b')\,y+(c+\lambda c')\,z+d+\lambda d'=0.$$

This being parallel to the second line, we have

$$(a+\lambda a')\,l+(b+\lambda b')\,m+(c+\lambda c')\,n=0,$$

i.e.,
$$\lambda=-\frac{al+bm+cn}{a'l+b'm+c'n}.$$

Hence the required equation of the plane is

$$(a'l+b'm+c'n)\,(ax+by+cz+d)$$
$$-(al+bm+cn)\,(a'x+b'y+c'z+d')=0.$$

3. Find the equation to the plane through the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ parallel to the line $\frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}$.

(*Rohilkhand*, 1977)

Ans. $\Sigma\,(x-\alpha)\,(mn'-nm')=0.$

4. Prove that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{3}=\frac{z-4}{3}$ are coplanar. (*Lucknow*, 1979)

5. Find the equation of the plane through $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ which is perpendicular to the plane containing $\frac{x}{m}=\frac{y}{n}=\frac{z}{l}$ and $\frac{x}{n}=\frac{y}{l}=\frac{z}{m}$.

(*Gorakhpur*, 1973; *Agra*, 1975)

Solution. Let $ax+by+cz=0$ be the equation of a plane containing the first line.

Then,
$$al+bm+cn=0 \qquad \ldots(1)$$

The equation to the plane containing the other two lines is

$$\begin{vmatrix} x & y & z \\ m & n & l \\ n & l & m \end{vmatrix}=0$$

or
$$x\,(mn-l^2)+y\,(ln-m^2)+z\,(lm-n^2)=0.$$

From the condition of the equation,

$$a\,(mn-l^2)+b\,(ln-m^2)+c\,(ml-n^2)=0. \qquad \ldots(2)$$

From (1) and (2), if $lm+mn+nl\neq 0$,

$$a:b:c=(m-n):(n-l):(l-m).$$

Hence the required equation is

$$(m-n)\,x+(n-l)\,y+(l-m)\,z=0.$$

6. Find the equations to the perpendicular from the origin to the line

$$x+4y+4z-27=0,\ 2x+2y+5z-21=0.$$

Find also the coordinates of the foot of the perpendicular.

(*Patna*, 1968)

Solution. The equation of a plane through the given line is

$$x+4y+4z-27+\lambda\,(2x+2y+5z-21)=0.$$

This passes through the origin if

$$\lambda=-\frac{9}{7}.$$

Hence the equation of the plane through the origin and the line is

$$11x-10y+17z=0 \qquad \ldots(1)$$

Now the direction cosines of the given line are proportional to l, m, n, where

$$l+4m+4n=0,\ 2l+2m+5n=0,$$

i.e.,
$$l:m:n=4:1:-2.$$

The equation of the plane through the origin perpendicular to the line is

$$4x+y-2z=0. \qquad \ldots(2)$$

The perpendicular is the line of intersection of (1) and (2). Its equations are, therefore,

$$\frac{x}{1}=\frac{y}{30}=\frac{z}{17}.$$

Let the foot of perpendicular be $(\lambda, 30\lambda, 17\lambda)$. Then this point must lie on either of the given planes. Taking the first plane,

$$\lambda+120\lambda+68\lambda-27=0. \quad \text{i.e., } \lambda=\frac{1}{7}.$$

Hence the coordinates of the foot of the perpendicular are

$$\left(\frac{1}{7}, \frac{37}{7}, \frac{17}{7}\right).$$

7. Show that the line through (α, β, γ) which is parallel to the plane $lx+my+nz=0$ and intersects the line

$$u \equiv ax+by+cz+d=0, \quad v \equiv a'x+b'y+c'z+d'=0,$$

is

$$l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$$

$$u(a'\alpha+b'\beta+c'\gamma+d')=v(a\alpha+b\beta+c\gamma+d).$$

Hint. The required line lies in the plane through (α, β, γ) parallel to $lx+my+nz=0$. It also lies in the plane $u+\lambda v=0$ where λ is so determined that the plane $u+\lambda v=0$ passes through (α, β, γ).

16.73 Shortest distance between two lines.

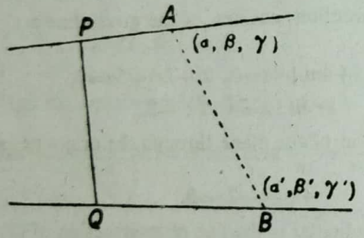
We shall now find the shortest distance between the lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

and

$$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}.$$

Let A be the point (α, β, γ) on the first line and B the point $(\alpha', \beta', \gamma')$ on the second line. The shortest distance PQ between the given lines is at right angles to both and it therefore equal to the projection of AB on PQ itself.



If λ, μ, υ be the direction cosines of PQ, the shortest distance is equal to

$$\lambda(\alpha-\alpha')+\mu(\beta-\beta')+\upsilon(\gamma-\gamma'). \qquad \ldots(1)$$

But $$l\lambda+m\mu+n\upsilon=0, \qquad \ldots(2)$$

and also $$l'\lambda+m'\mu+n'\upsilon=0. \qquad \ldots(3)$$

From (2) and (3),

$$\frac{\lambda}{mn'-nm'}=\frac{\mu}{nl'-ln'}=\frac{\upsilon}{lm'-ml'}.$$

Hence, from (1), the shortest distance between the given lines is

$$\frac{(\alpha-\alpha')(mn'-nm')+(\beta-\beta')(nl'-ln')+(\gamma-\gamma')(lm'-ml')}{\sqrt{\Sigma(mn'-nm')^2}},$$

$$=\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \sqrt{\Sigma(mn'-nm')^2}$$

Further, the equations to the planes APQ and BPQ are respectively

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ \lambda & \mu & \upsilon \end{vmatrix}=0, \quad \begin{vmatrix} x-\alpha' & y-\beta' & z-\gamma' \\ l' & m' & n' \\ \lambda & \mu & \upsilon \end{vmatrix}=0,$$

and these represent the line PQ.

Examples

1. Find the length and equations of the common perpendicular to the two lines

$$\frac{x+3}{-4}=\frac{y-6}{3}=\frac{z}{2}$$

and

$$\frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}.$$

(*Math. Tripos.*, 1952; *Gorakhpur*, 1974; *Ranchi*, 1971)

Solution. Let λ, μ, υ be the direction cosines of the common perpendicular. Then,

$$-4\lambda+3\mu+2\upsilon=0,$$

and

$$-4\lambda+\mu+\upsilon=0.$$

From these,

$$\frac{\lambda}{1}=\frac{\mu}{-4}=\frac{\upsilon}{8}$$

The length of the common perpendicular is

$$\frac{1.1+(-6)(-4)+7.8}{9}=9.$$

Its equations are

$$\begin{vmatrix} x+3 & y-6 & z \\ -4 & 3 & 2 \\ 1 & -4 & 8 \end{vmatrix}=0, \quad \begin{vmatrix} x+2 & y & z-7 \\ -4 & 1 & 1 \\ 1 & -4 & 8 \end{vmatrix}=0.$$

Aliter. The above example can also be worked out by the following method :

Let the common perpendicular meet the lines in P and Q respectively. Then the coordinates of P and Q may be written as $(-3-4r, 6+3r, 2r)$, $(-2-4r', r', 7+r')$ where r is proportional to the distance of P from $(-3, 6, 0)$, and r' to the distance of Q from $(-2, 0, 7)$.

Since PQ is perpendicular to both lines, we have

$$-4(-1-4r+4r')+3(6+3r-r')+2(2r-7-r')=0,$$

and

$$-4(-1-4r+4r')+1(6+3r-r')+1(2r-7-r')=0.$$

Solving these we get $r=-1$ and $r'=-1$.

Therefore P and Q are the points $(1, 3, -2)$, $(2, -1, 6)$.

The length of the common perpendicular PQ is therefore 9 and its equations are

$$\frac{x-1}{1}=\frac{y-3}{-4}=\frac{z+2}{8}.$$

2. Show that the shortest distance between the lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}\ ;\ \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$$

is $3\sqrt{30}$, and that its equations are

$$\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}.$$

(*U. P. C. S.*, 1970; *Lucknow*, 1980; *Meerut*, 1974; *Rajasthan*, 1976)

3. Find the magnitude and the equations of the line of shortest distance between the lines

$$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7};\ \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}.$$

(*Andhra Hons.*, 1966)

Ans. 14; $117x+4y-41z-490=0$,
$9x-4y-z-14=0$.

4. Show that the shortest distance between the axis of z and the line

$$ax+by+cz+d=0 \text{ and } a'x+b'y+c'z+d'=0$$

is

$$\frac{cd'-c'd}{\sqrt{(ac'-a'c)^2+(bc'-b'c)^2}} \qquad (\textit{Kanpur}, 1976)$$

Hint. The plane passing through the line and parallel to the axis of z is

$$c'(ax+by+cz+d)=c(a'x+b'y+c'z+d').$$

The shortest distance is the length of the perpendicular from the origin on this plane.

Show that the equation to the plane containing the line $\frac{y}{b}+\frac{z}{c}=1$, $x=0$, and parallel to the line $\frac{x}{a}-\frac{z}{c}=1$, $y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$, and if $2d$ is the shortest distance, prove that

$$\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

(*Gorakhpur*, 1973; *Agra*, 1977)

6. Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes $y+z=0$, $z+x=0$, $x+y=0$, $x+y+z=a$ is $\frac{2a}{\sqrt{6}}$ and that the three lines of shortest distance intersect at the point $x=y=z=-a$.

((*I. A. S.*, 1972; *Gorakhpur*, 1975)

16.74 Line intersecting two given lines. Let $\mu_1=0$, ${}_1\upsilon=0$; $\mu_2=0$, $\upsilon_2=9$ be two given lines.

The line $\mu_1+\lambda_1\upsilon_1=0$, $\mu_2\lambda_2\upsilon_2=0$ intersects each of the given lines for it is coplanar with each of them.

Ex. 1. Find the equation to the line that the intersect, the lines $x+y+z=0$, $2x+y+4z=1$; $x-y-z=1$, $6x+2y-z=3$, and passes through the point $(1, 1, 1)$.

Ans. $\frac{8x-5}{3}=\frac{y+1}{2}=\frac{z}{1}$.

Ex. 2. A line with direction cosines proportional to $2, 7, -5$ is drawn to intersect the lines

$$\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}\ ;\ \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}.$$

Find the coordinates of the points of intersection and the length intercepted on it. (*Lucknow*, 1979; *Gorakhpur*, 1968)

Solution. Let (α, β, γ), $(\alpha', \beta', \gamma')$ be the respective points of intersection with the given lines.

Then, $$\frac{\alpha-5}{3}=\frac{\beta-7}{-1}=\frac{\gamma+2}{1}=r, \text{ say;}$$

and $$\frac{\alpha'+3}{-3}=\frac{\beta'-3}{2}=\frac{\gamma'-6}{4}=r', \text{ say.}$$

Then $\alpha=5+3r, \beta=7-r, \gamma=-2+r;$

and $\alpha'=-3-2r', \beta'=3+2r', \gamma'=6+4r'.$

The direction cosines of the third line are proportional to

$$\alpha-\alpha', \beta-\beta', \gamma-\gamma'.$$

This gives $$8+3r+3r'=2k,$$
$$4-r-2r'=7k,$$
$$-8+r-4r'=-5k,$$

were k is a constant.

From these,
$$23r+25r'+48=0$$
$$17r+7r'+24=0.$$

Solving, $r=r'=-1.$

The points of intersection are thus $(2, 8, -3)$, $(0, 1, 2)$, and the required length of the intercept is $7\sqrt{8}$.

Ex. 3. Find the surface generated by a straight line which intersects the lines $x+y-z=0$, $x-y=z$, $x+y=2a$, and the parabola $y=0$, $x^2=2az$.

Ans. $x^2-y^2=2az$.

Hint. If the direction ratios of the given line are l, m, n; the equation of the line is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

Put $y=0$, solve for x and z, substitute in $x^2=2az$ and proceed.

16.75 Intersection of three planes.

Let
$$u_1\equiv a_1x+b_1y+c_1z+d_1=0,$$
$$u_2\equiv a_2x+b_2y+c_2z+d_2=0,$$
$$u_3\equiv a_3x+b_3y+c_3z+d_3=0,$$

be the equations of the three given planes. The planes may intersect in a finite point or they may have a common line of intersection. We shall first find the condition that there may be a **common line of intersection** of the given planes.

Any plane through the line or intersection $u_1=0$, $u_2=0$ is given by

$$\lambda_1u_1+\lambda_2u_2=0. \qquad ...(1)$$

where λ_1 and λ_2 are arbitrary constants.

If the planes $u_1=0$, $u_2=0$, $u_3=0$ have a common line of intersection, equation (1) must, for some values of λ_1, λ_2, represent the same plane as that represented by $u_3=0$. We thus have

$$\lambda_1u_1+\lambda_2u_2\equiv-\lambda_3u_3.$$

or $$\lambda_1u_1+\lambda_2u_2+\lambda_3u_3\equiv0. \qquad ...(2)$$

where λ_3 is a constant.

Conversely, if $\lambda_1, \lambda_2, \lambda_3$ can be found so that relation (2) is satisfied, we have

$$\lambda_1u_1+\lambda_2u_2=-\lambda_3u_3,$$

this is, the plane $u_3=0$ passes through the line of intersection of the planes $u_1=0$, $u_2=0$. Relation (2) being an identity, the coefficients of x, y, z and the constant term on the left hand side of (2) must separately vanish. We thus have

$$a_1\lambda_1+a_2\lambda_2+a_3\lambda_3=0.$$
$$b_1\lambda_1+b_2\lambda_2+b_3\lambda_3=0,$$
$$c_1\lambda_1+c_2\lambda_2+c_3\lambda_3=0,$$
and $$d_1\lambda_1+d_2\lambda_2+d_3\lambda_3=0.$$

We can take any three of the above equations to eliminate $\lambda_1, \lambda_2, \lambda_3$. This will, however, leave one equation unsatisfied. If we take the remaining fourth equation and eliminate $\lambda_1, \lambda_2, \lambda_3$ between this and any two equation taken earlier, we shall get one more condition in order that the three planes may have a common line of intersection.

The *condition for a common line of intersection* of the three planes can be expressed as

$$\left\|\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{matrix}\right\|=0. \qquad ...(3)$$

the double vertical lines signifying that any two of the four third order determinants are zero.

In some cases it is convenient to use relation (2) directly. It states that the planes $u_1=0$, $u_2=0$, $u_3=0$ have a common line of intersection if three constants $\lambda_1, \lambda_2, \lambda_3$ can be determined such that $\lambda_1u_1+\lambda_2u_2+\lambda_3u_3$ vanishes identically.

If the planes do not have a common line of intersection, the coordinates of their point of intersection can be obtained by solving their equations as simultaneous equations in x, y, and z.

Note. The reader should prove as an exercise that if any two of the four determinants in (3) vanish, the other two also vanish.

Ex. Show that the planes $x+y+z+1=0$.

$$2x-y+3z+5=0 \text{ and } 5x+2y+6z+8=0$$

have a common line of intersection.

Hint. $3(x+y+z+1)+(2x-y+3z+5)$
$$-(5x+2y+6z+8)=0.$$

16.76 Lines intersecting three lines.

Let the equations to three given lines be

$$u_1=0=v_1,\ u_2=0=v_2,\ u_3=0=v_3.$$

These lines lie respectively in the planes

$$u_1-\lambda_1 v_1=0, \qquad \ldots(1)$$
$$u_2-\lambda_2 v_2=0, \qquad \ldots(2)$$
and
$$u_3-\lambda_3 v_3=0. \qquad \ldots(3)$$

If the three given lines are intersected by a line, it must be coplanar with each of these, i.e., the above three planes must have a common line of intersection which is the line intersecting the three given lines.

Now, from the preceding article, we know that in order that three given planes may have a common line of intersection, two independent conditions must be satisfied. We have expressed these conditions in terms of a_1, b_1 etc. These could as well be expressed in terms of $\lambda_1, \lambda_2, \lambda_3$. Let the two conditions be written as

$$\left.\begin{aligned} &f_1(\lambda_1, \lambda_2, \lambda_3)=0,\\ \text{and}\quad &f_1(\lambda_1, \lambda_2, \lambda_3)=0.\end{aligned}\right\} \qquad \ldots(4)$$

If we can find the values of $\lambda_1, \lambda_2, \lambda_3$ satisfying equations (4), any two of the equations (1), (2) and (3) represent a line intersecting the three given lines. Let us, for example, choose equations (1) and (2). Eliminating λ_3 between the equations (4), we get

$$f(\lambda_1, \lambda_2)=0 \qquad \ldots(5)$$

which is satisfied by an infinite number of values of λ_1, λ_2. We thus get an infinite number of lines intersecting the three given lines.

If we wish to find the locus of all such lines, we eliminate λ_1, λ_2 between (1), (2), (5). The required locus, or the surface generated by such lines is thus

$$f\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}\right)=0.$$

The above process is equivalent to eliminating $\lambda_1, \lambda_2, \lambda_3$ from any three of the five equations (1), (2), (3), (4) and (5). The surface generated by lines intersecting the three given lines is thus also given by either of the equations

$$f_1\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}\right)=0,\ f_2\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}\right)=0.$$

Example. Find the locus of all lines which intersect the lines $y=mx$, $z=c$; $y=-mx$, $z=-c$; and the x-axis. (*Meerat*, 1973)

Solution. Let the three planes

$$y-mx+\lambda_1(z-c)=0,\ y+mx+\lambda_2(z+c)=0\ y+\lambda_3 z=0$$

have a common line of intersection; it meets the three given lines. That is so if

$$\begin{Vmatrix} -m & 1 & \lambda_1 & -\lambda_1 c \\ m & 1 & \lambda_2 & \lambda_2 c \\ 0 & 1 & \lambda_3 & 0 \end{Vmatrix}=0,$$

i.e., (1) $\lambda_3(\lambda_1-\lambda_2)=0$ and (2) $2\lambda_3=\lambda_1+\lambda_2$. Taking (1), we get $\lambda_1=\lambda_2$, since $\lambda_3\neq0$. The coordinates of any point on a line which meets the three given lines satisfy

$$y-mx+\lambda_1(z-c)=0,\ y+mx+\lambda_2(z+c)=0$$

where $\lambda_1=\lambda_2$.

Eliminating λ_1, λ_2, the required locus is

$$mxz=cy.$$

Note. The reader should verify that the same result is obtained from (2).

Examples

1. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$$

and the plane $x-y+z=5$. (*Agra*, 1972; *Gorakhpur*, 1972)

Ans. 13 units.

2. A variable plane makes intercepts on the coordinate axes the sum of whose squares is constant and is equal to k^2. Show that the locus of the root of the perpendicular from the origin to the plane is

$$(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=k^2. \qquad (\textit{Sagar}, 1968)$$

3. Find the equation of the line passing through the points (a_1, b_1, c_1) and (a_2, b_2, c_2), and prove that it passes through the origin if $a_1a_2+b_1b_2+c_1c_2=r_1r_2$, where r_1 and r_2 are the distances of these points from the origin. (*Rajasthan*, 1969; *Kanpur*, 1976)

4. Prove that the lines $x=ay+b$, $z=cy+d$ and $x=a'y+b'$, $z=c'y+d'$ are perpendicular if $aa'+cc'+1=0$.

(*Gorakhpur*, 1970; *Delhi*, 1971)

5. Find the equation of the plane containing the line

$$\frac{x-2}{2}=\frac{y-3}{3}=\frac{z-4}{5}$$

and parallel to x-axis. (*Agra*, 1974)

Ans. $5y-3z=3$.

6. Prove that the equation of the plane through the point (α, β, γ) and containing the line $x=py+q=rz+s$, is

$$\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix}=0.$$

(*Meerut*, 1969)

7. Show that the equation of the plane containing the lines

$$ax+by+cz=0=a'x+b'y+c'z$$

and

$$\alpha x+\beta y+\gamma z=0=\alpha'x+\beta'y+\gamma'z$$

is

$$\begin{vmatrix} x & y & z \\ bc'-b'c & ca'-c'a & ab'-a'b \\ \beta\gamma'-\beta'\gamma & \gamma\alpha'-\gamma'\alpha & \alpha\beta'-\alpha'\beta \end{vmatrix}=0.$$

(*Meerut*, 1972; *I. A. S.*, 1977)

Hint. The direction cosines of two lines are proportional to $bc'-b'c$, $ca'-c'a$, $ab'-a'b$ and $\beta\gamma'-\beta'\gamma$, $\gamma\alpha'-\gamma'\alpha$, $\alpha\beta'-\alpha'\beta$. The plane

$$Ax+By-Cz=0,$$

contains the two lines if

$$A(bc'-b'c)+B(ca'-c'a)+C(ab'-a'b)=0$$

and

$$A(B\gamma'-\beta'\gamma)+B(\gamma\alpha'-\gamma'\alpha)+C(\alpha\beta'-\alpha'\beta)=0,$$

Eliminating A, B, C.

8. Find the length of perpendicular from the point $(1, -2, 3)$ on a line through $(2, -3, 5)$, making equal angle with coordinate axes.

(*Gorakhpur*, 1974)

Ans. $\sqrt{14/3}$.

9. If P is the point $(1, 2, 3)$ and the equation of the line AB is

$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5},$$

find the equation of the line PN as intersection of two planes where N is the foot of the perpendicular from P on AB. Also find the coordinates of N. (*Meerut*, 1976)

Ans. $x-2y+z=0=3x+4y+5z-26$; $\left(\frac{32}{25}, \frac{51}{25}, \frac{70}{25}\right)$.

10. Prove that the lines

$$\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta}$$

and

$$\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$$

are coplanar and find the equation to the plane containing them.

(*Jodhpur*, 1968)

Ans. $x-2y+z=0$.

11. Show that the lines $x+y+z-3=0=2x+3y+4z-5$ and $4x-y+5z-7=0=2x-5y-z-3$ are coplanar and find the equation of the plane in which they lie. (*Agra*, 1975)

Ans. $x+2y+3z-2=0$.

12. Prove that the planes $x+y+z+6=0$, $x+2y+2z+6=0$ and $x+3y+3z+6=0$ intersect in one common line.

(*Rajasthan*, 1976)

13. Find the equations of the two planes through the origin which are parallel to the line $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z-1}{-2}$ and at a distance $(5/3)$ units from it. (*Meerut*, 1976)

Ans. $x-2y+2z=0$, and $2x+2y+z=0$.

14. Find the length of the short distance between the lines

$$\frac{x-3}{1}=\frac{y-5}{-10}=\frac{z-7}{7},$$

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$$

(*Meerut*, 1977)

Also find its equation.

Ans. $2\sqrt{29}$; $\frac{x-3}{2}=\frac{y-5}{3}=\frac{z-7}{4}$

15. Prove that the shortest distances between the diagonals of a rectangular parallelopiped and the edges not meeting it are

$$\frac{bc}{\sqrt{(b^2+c^2)}}, \frac{ca}{\sqrt{(c^2+a^2)}}, \frac{ab}{\sqrt{(a^2+b^2)}},$$

where a, b, c are the lengths of the edges.

16. A variable straight line intersects x-axis and the curve $x=y$, $y^2=cz$, and is parallel to the plane $x=0$. Prove that the line generates the surface $xy=cz$. (*Meerut* 1976; *Kanpur*, 1977)

Solution. The equation of the plane parallel to $x=0$ is

$$x=\lambda, \qquad \text{...(1)}$$

and the equation of the plane containing x-axis is

$$y=\mu z. \qquad \text{...(2)}$$

The required line is the intersection of (1) and (2). If it meets the curve $x=y, y^2=cz$, we have

$$x=\lambda=y \text{ and } \mu=\frac{y}{z}=\frac{c}{y} \text{ from } y^2=cz.$$

Consequently $\lambda\mu=c.$ ...(3)

Putting the values of λ and μ from (1) and (2) in (3), we obtain

$$x\left(\frac{y}{z}\right)=c$$

or

$$xy=cz.$$

17. Find the surface generated by a line which intersects the lines $y=a=z$, $x+3z=a=y+z$ and is parallel to the plane $x+y=0$. (*Kanpur*, 1975)

Solution. Any plane passing through the intersection of the planes $y=a=z$, $x+3z=a=y+z$ are

$$y-a+\lambda_1(z-a)=0 \text{ and } (x+3z-a)+\lambda_2(y+z-a)=0$$

or

$$y+\lambda_1 z-a(1+\lambda_1)=0 \text{ and } x+\lambda_2 y+z(3+\lambda_2)-a(1+\lambda_2)=0.$$

If l, m, n be the direction cosines of the required line, then

$$m+n\lambda_1=0 \text{ and } l+m\lambda_2+n(3+\lambda_2)=0.$$

These equations give

$$\frac{l}{3+\lambda_2-\lambda_1\lambda_2}=\frac{m}{\lambda_1}=\frac{n}{-1}.$$

If this line is parallel to the plane $x+y=0$, then

$$(3+\lambda_2-\lambda_1\lambda_2)+\lambda_1=0$$

or

$$\lambda_1+\lambda_2-\lambda_1\lambda_2+3=0.$$

Putting the values of λ_1 and λ_2, we obtain

$$\frac{a-y}{z-a}+\frac{a-x-3z}{y+z-a}+\left(\frac{a-y}{z-a}\right)\left(\frac{a-x-3z}{y+z-a}\right)+3=0.$$

This gives on simplifying

$$(y+z)(x+y)=2a(x+z),$$

which is the equation of the required surface.

18. Prove that a variable line which intersects the three given lines $y=mx$, $z=c$; $y=-mx$, $z=-c$ and $y=z$, $mx+c=0$ generates the surface

$$y^2-m^2x^2=z^2-c^2. \qquad \text{(Meerut, 1974)}$$

Solution. The equation of planes passing through the intersection of given pair of planes are

$$y-mx+\lambda_1(z-c)=0, \qquad \text{...(1)}$$

and

$$y+mx+\lambda_2(z+c)=0. \qquad \text{...(2)}$$

The intersection of (1) and (2) is the required line. Writing c for $-mx$ and y for z, in the above,

$$(y+c)+\lambda_1(y-c)=0$$

giving

$$\frac{y+c}{y-c}=\lambda_1, \qquad \text{...(3)}$$

and

$$(y-c)+\lambda_2(y+c)=0$$

giving

$$\frac{y-c}{y+c}=-\lambda_2. \qquad \text{...(4)}$$

From (3) and (4), $\lambda_1\lambda_2=1.$ ...(5)

Put the values of λ_1 and λ_2 from (1) and (2) in (5). The result follows immediately.

16.8 The sphere. *The sphere is the locus of a point in space which moves such that its distance from a fixed point always remains the same.* The fixed point is called the *centre* and the constant distance the *radius* of the sphere.

If (α, β, γ) be the centre of a sphere and r its radius, the equation to the sphere can easily be seen to be

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2.$$

Now, and equation of the form $ax^2+ay^2+az^2+2ux+2vy+2wz+d=0$ can be written as

$$\left(x+\frac{u}{a}\right)^2+\left(y+\frac{v}{a}\right)^2+\left(z+\frac{w}{a}\right)^2=\frac{u^2+v^2+w^2-ad}{a^2},$$

and therefore represents a sphere whose centre is

$$\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right) \text{ and radius } \frac{\sqrt{(u^2+v^2+w^2-ad)}}{a}$$

Corollary. The equation of the sphere whose centre is the origin and radius a, is

$$x^2+y^2+z^2=a^2.$$

16.81 Equation of the sphere on the join of (x_1, y_1, z_1), (x_2, y_2, z_2), as diameter.

If P be any point (x, y, z) on the sphere, the direction cosines of the lines joing P to the extremities of the given diameter are proportional to $(x-x_1)$, $(y-y_1)$, $(z-z_1)$, and $(x-x_2)$, $(y-y_2)$, $(z-z_2)$.

The two lines are at right angles. Therefore,

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

is the equation of the sphere.

Examples

1. Find the equation of the sphere which passes through the origin and the points (2, 1, −1), (1, 5, −4), (−2, 4, −6).

Ans. $x^2+y^2+z^2+2x-4y+6z=0.$

2. Find the equation of the sphere which passes through the points (1, −3, 4), (1, −5, 2) and (1, −3, 0) and whose centre lies on the plane $x+y+z=0$. (*Lucknow*, 1960)

Ans. $(x-1)^2+(y+3)^2+(z-2)^2=4.$

3. Find the equation of the sphere passing through the circle of intersection of the spheres

$$x^2+y^2+z^2-2x+4y=11,$$

and

$$x^2+y^2+z^2-4x-6z=51,$$

and the point (−7, 4, 12). (*Lucknow*, 1955)

Ans. $x^2+y^2+z^2-6x-4y-12z-91=0.$

4. Find the equations to the sphere through the circle $x^2+y^2+z^2=9$, $x+y-2z=4$ and (i) the origin, (ii) the point (1, 2, −1). (*Lucknow*, 1978)

Ans. (i) $4(x^2+y^2+z^2)-9x-9y+18z=0;$
(ii) $x^2+y^2+z^2+3x+3y-6z-21=0.$

5. The equation of the plane ABC is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Obtain the equations of the circle passing through A, B and C, where A, B and C are the points of intersection of the plane with the coordinates axes. (*Gorakhpur*, 1973; *Lucknow*, 1968)

Ans. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1;\ x^2+y^2+z^2-ax-by-cz=0$

6. Prove that the circles

$$x^2+y^2+z^2-2x+3y+4z-5=0,\ 5y+6z+1=0$$

and

$$x^2+y^2+z^2-3x-4y+5z-6=0,\ x+2y-7z=0$$

lie on the same sphere, and find its equation. (*Lucknow*, 1961)

Solution. The equations

$$x^2+y^2+z^2-2x+3y+4z-5+\lambda(5y+6z+1)=0$$

and

$$x^2+y^2+z^2-3x-4y+5z-6+\mu(x+2y-7z)=0$$

must be the same. Comparing coefficients, we find that $\lambda=-1$ and $\mu=1$. Hence the required equation of the sphere is

$$x^2+y^2+z^2-2x-2y-2z-6=0.$$

7. A plane passes through a point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere $OABC$ is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2.$$

(*U. P. C. S.*, 1970; 74, 77; *Lucknow*, 1980; *Rajasthan*, 1976)

Hint. If the distances OA, OB, OC are 2α, 2β, 2γ, the centre of the sphere $OABC$ is (α, β, γ).

8. Show that the radius of the circle

$$x^2+y^2+z^2+x+y+z-4=0,\ x+y+z=0 \text{ is } 2.$$

(*Gorakhpur*, 1969)

16.82 Euation of tangent plane. Let the sphere be $x^2+y^2+z^2=a^2$ and let (x', y', z') be any point on it. The direction cosines of the line joining this point to the centre of the sphere are proportional to x', y', z'.

Now the tangent plane at (x', y', z') is perpendicular to the join of the centre and this point. Its equation is therefore

$$xx'+yy'+zz'=x'^2+y'^2+z'^2$$
$$=a^2,$$

since (x', y', z') is a point of the sphere.

Examples

1. Show that the equation of the tangent plane at a point (x', y', z') of the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is

$$xx'+yy'+zz'+u(x+x')+v(y+y')+w(z+z')+d=0.$$

2. Find the condition that the plane $lx+my+nz=p$ should touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

Ans. $(ul+vm+wn+p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d)$.

3. Show that the spheres

$$x^2+y^2+z^2=25$$

and $$x^2+y^2+z^2-24x-40y-18+225=0$$

touch externally. (*Roorkee*, 1957)

4. Find the equations of spheres whose two tangent planes are $x-2z=8$ and $2x-z+5=0$ and whose centres lie on the straight line $x=-2, y=0$. (*Lucknow*, 1959)

Ans. $5(x^2+y^2+z^2)+20x+110z+481=0,$
$5(x^2+y^2+z^2)+20x+30z+49=0.$

5. Find the equations to the spheres which pass through the circle $x^2+y^2+z^2=5$, $x+2y+3z=3$, and touch the plane $4x+3y=15$.

(*Gorakhpur*, 1972; *Kurukshetra*, 1976)

Ans. $x^2+y^2+z^2+2x+4y+6z-11=0,$
$5(x^2+y^2+z^2)-4x-8y-12z-13=0$

6. Find the condition that the spheres whose equations are

$$x^2+y^2+z^2+2ax+2by+2cz+d=0,$$

and $$x^2+y^2+z^2+2Ax+2By+2Cz+D=0,$$

cut one another at right angles. (*Delhi*, 1977; *I. A. S.*, 1977)

Ans. $2aA+2bB+2cC=d+D$.

7. Prove that the centres of spheres which touch the lines $y=mx, z=c, y=-mx, z=-c$ lie on the conicoid

$$mxy+cz(1+m^2)=0.$$

(*Gorakhpur*, 1972; *Rajasthan*, 1968)

Solution. The equations to the lines can be written as

$$\frac{x}{1}=\frac{y}{m}=\frac{z-c}{0};\quad \frac{x}{1}=\frac{y}{-m}=\frac{z+c}{0}.$$

If (α, β, γ) be the centre of a sphere which touches the above lines, the length of the perpendicular from this point on each line must be the same. Therefore,

$$\alpha^2+\beta^2+(\gamma-c)^2-\frac{(\alpha+m\beta)^2}{1+m^2}$$
$$=\alpha^2+\beta^2+(\gamma+c)^2-\frac{(\alpha-m\beta)^2}{1+m^2}$$

i.e., $$m\alpha\beta+c\gamma(1+m^2)=0.$$

Hence the required locus is

$$mxy+cz(1+m^2)=0.$$

16.83 Power of a given point with respect to sphere.

Definition. If any secant through a given point O meets a given sphere in P and Q, then the measure of $OP.OQ$ is called the **power** of O with respect to the sphere.

We shall show that $OP.OQ$ is the same for all secants through O.

Let O be the point (α, β, γ), and let

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\ (=r)$$

be a secant with direction cosines l, m, n through O.

This meets the sphere

$$f(x, y, z)\equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

in P and Q where OP and OQ are the roots of the quadratic in r,

i.e., $$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2$$
$$+2u(\alpha+lr+2v)(\beta+mr)+2w(\gamma+nr)+d=0,$$

or $$r^2+2r\{l(\alpha+u)+m(\beta+v)+n(\gamma+w)\}+f(\alpha, \beta, \gamma)=0.$$

From this, $OP.OQ=f(\alpha, \beta, \gamma)$ which depends only on the position of O.

16.84 Radical plane of two spheres. *Definition.* The radical plane of the spheres is the locus of points whose powers with respect to the sphere are equal.

Let the spheres be

$$S_1\equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,$$
$$S_2\equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0.$$

The radical plane of $S_1=0$, $S_2=0$ is obviously

$$S_1-S_2=0,$$

or $$2\,(\mathbf{u}_1-\mathbf{u}_2)\,\mathbf{x}+2\,(\mathbf{v}_1-\mathbf{v}_2)\,\mathbf{y}+(\mathbf{w}_1-\mathbf{w}_2)\,\mathbf{z}+\mathbf{d}_1-\mathbf{d}_2=0.$$

The following results are easy to establish :

(*i*) *The radical plane of two spheres is at right angles to the line joining their centres.*

(*ii*) *The radical planes of three spheres taken two by two pass through one line.*

(The line is $S_1=S_2=S_3$).

(*iii*) *The radical planes of four spheres taken two by two pass through one point.*

16.85 Coaxal spheres. *Definition.* Spheres any two of which have the same radical planes are said to form coaxal system.

If we take the line joining the centres of two spheres as x-axis and the radical planes as $x=0$, the equations of the sphere can be written as

$$x^2+y^2+z^2+2\lambda_1 x+d=0$$

and $$x^2+y^2+z^2+2\lambda_2 x+d=0.$$

It is now easy to see that the equation

$$x^2+y^2+z^2+2\lambda x+d=0$$

where λ is a parameter, represents a *coaxal system* of spheres.

Examples

1. Find the equation to the sphere passing through the points $(0, 0, 0)$, $(0, 1, -1)$, $(-1, 2, 0)$ and $(1, 2, 3)$. (*Meerut*, 1975)

Ans. $7\,(x^2+y^2+z^2)-15x-25y-11z=0.$

2. Obtain the equation of the sphere having its centre on the line $5y+2z=0=2x-3y$ and passing through the points $(0, -2, -4)$ and $(2, -1, -1)$. (*Agra*, 1977)

Ans. $x^2+y^2+z^2-6x-4y+10z+12=0.$

3. Prove that the plane $x+2y-z=4$ cuts the sphere

$$x^2+y^2+z^2-x+z+2=0$$

in a circle of radius unity. Also find the equation of the sphere which has this circle for one of its great circles. (*Gorakhpur*, 1975)

Ans. $x^2+y^2+z^2-2x-2y+2z+2=0.$

4. Find the equation to the sphere through the circle $x^2+y^2+z^2=1$, $2x+4y+5z=6$ and touching the plane $z=0$. (*Meerut*, 1977)

Ans. $5\,(x^2+y^2+z^2-2x-4y-5z+1)=0.$

5. Find the equation to the sphere which passes through the point (α, β, γ) and the circle $x^2+y^2=a^2$, $z=0$. (*Meerut*, 1971; *Kanpur*, 1977)

Ans. $\gamma\,(x^2+y^2+z^2+a^2)+z\,(a^2-\alpha^2-\beta^2-\gamma^2)=0.$

6. If any tangent plane to the sphere $x^2+y^2+z^2=r^2$ makes intercepts a, b, c on the coordinate axes, show that

$$a^{-2}+b^{-2}+c^{-2}=r^{-2}.$$ (*Kanpur*, 1973; *Agra*, 1975)

7. Find the equation of tangent planes to the sphere

$$x^2+y^2+z^2+2x-4y+6z-7=0$$

which intersect in the line $6x-3y-23=0=3z+2$. (*Rohilkhand*, 1977)

Ans. $2x-y+4z=5$, $4x-2y-z=16$.

8. Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1, -2, 1)$ and cuts the sphere $x^2+y^2+z^2-4x+6y+4=0$ orthogonally. (*Gorakhpur*, 1967)

Ans. $x^2+y^2+z^2+7x+10y-5z+12=0.$

9. Two spheres of radii r_1 and r_2 intersect orthogonally. Prove that the radius of the common circle is

$$\frac{r_1 r_2}{\sqrt{(r_1^2+r_2^2)}}$$ (*Kanpur*, 1977)

10. Prove that a sphere which cuts the sphere $S_1=0$ and $S_2=0$ orthogonally, will also cut the sphere $\lambda_1 S_1+\lambda_2 S_2=0$ orthogonally for all values of $\lambda_1\,\lambda_2$. (*Rajasthan*, 1975)

11. Obtain the equation of the radical axis for the spheres

$$(x-2)^2+y^2+z^2=1;\ x^2+(y-3)^2+z^2=6$$

and $$(x+2)^2+(y+1)^2+(z-2)^2=6.$$ (*Kanpur*, 1971)

Ans. $x/3=y/2=z/7$.

12. Prove that the sum of squares of the intercepts by a given sphere on any three mutually perpendicular lines through a fixed point is constant. (*Gorakhpur*, 1971; *I. A. S.*, 1969)

13. A variable sphere passes through the points $(0, 0, \pm c)$ and cuts the lines $y=x\tan\alpha$, $z=c$; $y=-x\tan\alpha$, $z=y-c$ in the points P and P'. If PP' has constant length $2a$, show that the centre of the sphere lies on the circle $z=0$, $x^2+y^2=(a^2-c^2)\operatorname{cosec}^2 2\alpha$. (*Kanpur*, 1974; *Rajasthan*, 1975)

14. Prove that the locus of points whose powers with respect to two given spheres are in a constant ratio is a sphere coaxal with the two given spheres.

15. A system of spheres is represented by the equation $x^2+y^2+z^2+2\mu y+2\mu' z-d=0$, where μ and μ' are parameters. Show that this system passes through the limiting points of the coaxal system $x^2+y^2+z^2+2\lambda x+d=0$; and cuts every member of the coaxal system at right angles.

Hint. The limiting points of the coaxal system are

$$(\pm\sqrt{d}, 0, 0).$$

Examples on Chapter XVI

1. Find the equation of the plane passing through the line of intersection of the planes $2x+y+3z=0$, $x-2y=0$ and perpendicular to the plane $3x+y-2z=0$.

Ans. $x+3y+3z=0$.

2. Find the equation of the plane which passes through the axis of and is perpendicular to the line.

$$\frac{x-1}{\cos\theta}=\frac{y+2}{\sin\theta}=\frac{z-3}{0}$$ (*Lucknow*, 1980)

Ans. $x+y\tan\theta=0$.

3. A variable plane passes through a fixed point (a, b, c) and meets the coordinate axes in A, B, C. Show that the locus of the point of intersection of the planes through A, B, C parallel to the coordinate planes is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1.$$ (*Patna*, 1967)

4. Prove that the lines

$$\frac{x-3}{1}=\frac{y+4}{-3}=\frac{z-5}{3},$$

and

$$\frac{x-4}{1}=\frac{y-5}{3}=\frac{z-6}{-4}$$

intersect, and find the coordinates of the intersection.

Ans. $(2, -1, 2)$.

5. Show that the straight lines whose direction cosines are given by the equation $ul+vm=0$ and $al^2+bm^2+cn^2=0$ are

(*a*) perpendicular if

$$u^2(b+c)+v^2(c+a)+w^2(a+b)=0, \text{ and}$$

(*b*) parallel if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=0.$

(*Rajasthan*, 1970; *Agra*, 1974)

6. Show that the lines whose direction cosines are given by $l+m+n=0$ and $2mn+3nl-5lm=0$ are perpendicular to one another. (*Kurukchetra*, 1974)

7. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two cocurrent straight lines, show that the direction cosines of the two lines bisecting the angles between them are proportional to $l_1 \pm l_2$, $m_1 \pm m_2$, $n_1 \pm n_2$. (*Rajasthan*, 1969; *Agra*, 1971)

8. The direction cosines of a variable line in two adjacent positions are l, m, n and $l+\delta l$, $m+\delta m$, $n+\delta n$; show that the small angle $\delta\theta$ between the two positions is given by

$$(\delta\theta)^2=(\delta l)^2+(\delta m)^2+(\delta n)^2.$$

(*Rajasthan*, 1971; *Meerut*, 1972; *Agra*, 1973; *Kanpur*, 1976)

9. Find the equation of the plane through the line

$$ax+by+cz+d=0=a'x+b'y+c'z+d'=0$$

and parallel to the line $x/l=y/m=z/n$. (*Meerut*, 1973)

Ans. $(ax+by)+(z+d)(a'l+b'm+c'n)$
$=(a'x+b'y+c'z+d')(al+bm+cn)$.

10. Find the equation of the straight line drawn from the origin to intersect the lines

$$3x+2y+4z-5=0=2x-3y+4z-1$$

$$2x-4y+z+6=0=3x-4y+z-3.$$

Ans. $\frac{x}{249}=\frac{y}{153}=\frac{z}{-52}$.

11. Find the equation to the perpendicular from the origin to the line

$$x+2y+3z+4=0=2x+3y+4z+5.$$

Also find the coordinates of the foot of the perpendicular.

Ans. $\left(\frac{x}{2}=\frac{y}{-1}=\frac{z}{-4}\right), \left(\frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}\right)$.

21. Prove that the three planes

$$by-ax=n,\ cx-bz=l,\ az-cy=m$$

pass through one line if $al+bm+cn=0$.

Find the direction ratios of the line.

Ans. b, a, c.

22. Prove that the locus of the line which intersects the three lines $y-z=1,\ x=0;\ z-x=1,\ y=0;\ x-y=1,\ z=0$ is

$$x^2+y^2+z^2-2yz-2xz-2xy=1.$$

23. Prove that the locus of a line which meets the lines

$y=\pm mx,\ z=\pm c$, and the circle $x^2+y^2=a^2\ z=0$ is

$$c^2m^2\,(cy-mxz)^2+c^2\,(zy-cmx)^2=a^2m^2\,(z^2-c^2).$$

Hint. The line $y-mx+\lambda\,(z-c)=0,\ y+mx+\mu\,(z+c)=0$ meets the given circle if

$$c^2\,[(\lambda+\mu)^2+m^2\,(\lambda-\mu)^2]=4a^2m^2.$$

Eliminate λ and μ between this equation and the equation of the line.

24. Obtain the equations of the spheres which pass through the points $(4, 1, 0), (2, -3, 4), (1, 0, 0)$ and touch the plane

$$2x+2y-z=11.$$

Ans. $x^2+y^2+z^2-6x+2y-4z+5=0.$

$16\,(x^2+y^2+z^2)-102x+50y-49z+86=0.$

25. A point moves so that the sum of the squares of its distances from the six faces of a cube is constant. Show that its locus is a sphere whose centre coincides with the centre of the cube.

(*U. P. C. S.*, 1968)

16. A sphere whose radius is a constant $(=k)$ passes through the origin. If the sphere cuts the coordinate axes in A, B, C; prove that the locus of the centroid of the triangle ABC is the sphere

$$9\,(x^2+y^2+z^2)=4k^2.$$

(*Kurukshetra*, 1974; *U. P. C. S.*, 1973)

27. P is a variable point on a given line and A, B, C are the projection on the axes. Show that the sphere $OABC$ passes through a fixed circle.

(*Lucknow*, 1971)

28. A circle, centre $(2, 3, 0)$ and radius 1, is drawn in the plane $z=0$. Find the equation of the sphere which passes through this circle and through the point $(1, 1, 1)$.

(*Lucknow*, 1979; *I. A. S.*, 1966)

Ans. $x^2+y^2+z^2-4x-6y-5z+12=0.$

29. Show that two circles

$$x^2+y^2+z^2-y+2z=0,\ x-y+z-2=0$$

and $\quad x^2+y^2+z^2+x-3y+z-5=0,\ 2x-y+4z-1=0$

lie on the same sphere and find its equation. (*Lucknow*, 1979)

Ans. $x^2+y^2+z^2+3x-4y+5z-6=0$

30. Find the equation of the sphere with smallest radius and which passes through the points $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$.

Ans. $3\,(x^2+y^2+z^2)-2\,(x+y+z)-1=0.$

31. Find the limiting points of the coaxal system of sphere determined by

$$x^2+y^2+z^2+3x-3y+6=0$$

and $\quad x^2+y^2+z^2-6y-6z+6=0.$

Ans. $(-1, 2, 1)\ (-2, 1, -1)$.

32. A sphere of constant radius r passes through the origin O and cuts the coordinate axes in A, B, C. Prove that the locus of the foot of the perpendicular from O to the plane ABC is

$$(x^2+y^2+z^2)^2\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)=4r^2.$$

(*Meerut*, 1973)

33. A variable plane is drawn parallel to the given plane $x/a+y/b+z/c=0$ and meets the coordinate axes in A, B, C. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

(*U. P. C. S.*, 1972; *Meerut*, 1974; *Agra*, 1975; *Rajasthan*, 1976; *I. A. S.*, 1972)

Solution. Let the variable plane be

$$\frac{x}{ak}+\frac{y}{bk}+\frac{z}{ck}=1 \qquad ...(1)$$

where k is arbitrary.

The circle ABC lies in the above plane and also on the sphere $OABC$ whose equation is

$$x^2+y^2+z^2-k\,(ax+by+cz)=0. \qquad ...(2)$$

Eliminating k between (1) and (2)

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

or

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

34. Find the equation of the sphere which passes through the point ((2, 3, 6) and the feet of the perpendiculars from this point on the coordinate planes.

Also find the equation of tangent planes to this sphere which are parallel to the plane $2x+2y+2z=1$, and find the coordinates of the point of contact. (*U. P. C. S.*, 1967)

Ans. Sphere : $x^2+y^2+z^2-2x-3y-6z=0$,

Planes : $4x+4y+2z+5=0$, $4x+4y+2z=37$,

Points of contact :

$$\left(\frac{10}{3}, \frac{23}{6}, \frac{25}{6}\right)\left(\frac{-61}{37}, \frac{-85}{74}, \frac{62}{37}\right)$$

35. Show that the coordinates of the circumcentre of the triangle ABC, where $A : (a, 0, 0)$ $B : (0, b, 0)$ and $C : (0, 0, c)$ are

$$\left[\frac{a(b^{-2}+c^{-2})}{3(a^{-2}+b^{-2}+c^{-2})}, \frac{a(c^{-2}+a^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}, \frac{c(a^{-2}+b^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}\right]$$

36. Prove that the straight lines whose direction cosines are given by the equation $al+bm+cn=0$ and $fmn+gnl+hlm=0$ are perpendicular if $(f/a)+(g/b)+(h/c)=0$ and parallel if $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch}=0$. (*Bihar*, 1970; *Kanpur*, 1973)

37. Find the equation of the line through $(f, g, h,)$, which is parallel to the plane $lx+my+nz=0$ and intersects the straight line

$$ax+by+cz+d=0=a'x+b'y+c'z+d'=0.$$

(*U. P. C. S.*, 1976)

Ans. $lx+my+nz=fl+gm+hn$;

$$\left(\frac{ax+by+cz+d}{af+bg+ch+d}=\frac{a'x+b'y+c'z+d'}{a'f+bg+c'h+d'}\right).$$

CHAPTER XVII

THE CYLINDER AND CONE

17.1 The Cylinder. *Definition.* A surface generated by a straight line which is parallel to a fixed straight line and intersects a given curve or touches a given surface is called a *cylinder.*

The fixed straight line is called the **axis** and the given curve, the **guiding curve** of the cylinder. The moving straight line in its different positions gives different *generators* of the cylinder.

17.2 Equation of a cylinder. *We shall now find the equation of the cylinder whose generators are parallel to the straight line*

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

and intersect that conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0, z=0.$$

Let (α, β, γ) be any point on the cylinder. The equation to a generator then is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

This meets the plane $z=0$ in the point

$$\left(\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0\right).$$

Since the generator intersects the given conic the coordinates of the above point must satisfy the equation of the conic.

We must have

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2$$
$$+2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{m\gamma}{n}\right)+c=0.$$

or $$a(n\alpha-l\gamma)^2+2h(n\alpha-l\gamma)(n\beta-m\gamma)$$
$$+b(n\beta-m\gamma)^2+2gn(n\alpha-l\gamma)+2fn(n\beta-m\gamma)+cn^2=0.$$

Hence the locus of (α, β, γ) or the required equation of the cylinder is

$$a(nx-lz)^2+2h(nx-lz)(ny-mz)+b(ny-mz)^2$$
$$+2gn(nx-lz)+2fn(ny-mz)+cn^2=0.$$

17.3 Right circular cylinder. *A right circular cylinder is the surface generated by a straight line which is at a constant distance from a fixed straight line and parallel to it.*

The constant distance is called the *radius* of the cylinder and the fixed straight line its *axis*.

13.31 Equation of a right circular cylinder.

Let the axis of the cylinder be the straight line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(1)$$

and let r be its radius.

If (x', y', z') be any point on the cylinder, its distance from the axis equals r.

Now, from § 16.5, the square of the distance of $(x'\ y', z')$ from (1) is

$$[\{(y'-\beta)\,n-(z'-\gamma)\,m\}^2+\{(z'-\gamma)\,l-(x'-\alpha)\,n\}^2$$
$$+\{(x'-\alpha)\,m-(y'-\beta)\,l\}^2]\div(l^2+m^2+n^2).$$

Hence the required equation of the cylinder is

$$\{(y-\beta)\,n-(z-\gamma)\,m\}^2+\{(z-\gamma)\,l-(x-\alpha)\,n\}^2$$
$$+\{(x-\alpha)\,m-(y-\beta)\,l\}^2=r^2\,(l^2+m^2+n^2).$$

Solved Example. Find the equation of the right circular cylinder of a radius 2 whose axis is the line

$$\tfrac{1}{2}(x-1)=(y-2)=\tfrac{1}{2}(z-3).$$

(*Lucknow* 1969; *Gorakhpur*, 1973)

Solution. The square of the distance of any point (x, y, z), lying on the cylinder, from its axis is

$$\{(x-1).1-(y-2).2\}^2+\{(y-2).2-(z-3).\ 1\}^2$$
$$+\{(z-3).2-(x-1).2\}^2\div(4+1+4)$$

i.e., $$(x-2y+3)^2+(2y-z-1)^2+(2z-2x-4)^2\div 9,$$

i.e., $$5x^2+8y^2+5z^2-4xy-4yz-8zx+22x-16y-14z+26\div 9.$$

This is equal to the square of the radius of cylinder.

Hence, $5x^2+8y^2+5z^2-4xy-4yz-8zx+22x-16y-14z+26=9.4,$
i.e., the equation to the cylinder is

$$5x^2+8y^2+5z^2-4xy-4yz-8zx+22x-16y-14z-10=0.$$

Examples

1. Find the equation of the cylinder whose generators are parallel to the line $x/1=y/(-2)=z/3$ and passing through the curve $x^2+2y^2=1,\ z=0.$

(*Lucknow*, 1979; *Gorakhpur*, 1974)

Ans. $3x^2+6y^2+3z^2-2xz+8yz-3=0.$

2. Find the equation of the right circular cylinder of radius 3 units and having for its axis the line

$$\frac{x-1}{2}=\frac{y-3}{2}=\frac{z-5}{-1}$$

(*Agra*, 1974)

Ans. $5x^2+5y^2+8z^2-8xy+4yz+4zx-6x-42y-96z+225=0.$

3. Find the equation of the right circular cylinder of radius 2 units whose axis passes through (1, 2, 3) and has direction cosines proportional to 2, −3, 6. (*Delhi*, 1971; *Agra*, 1972; *Meerut*, 1974)

Ans. $45x^2+40y^2+13z^2+36yz-24zx+12xy-42x$
$-496y-234z+1374=0.$

Imp

4. Show that the equation of the right circular cylinder described on the circle through the points $A:(1, 0, 0)$, $B:(0, 1, 0)$ and $C:(0, 0, 1)$ as the guiding curve is

$$x^2+y^2+z^2-yz-zx-xy=1.$$

(*Jodhpur*, 1970; *Gorakhpur*, 1975)

5. Show that the enveloping cylinder of the sphere $x^2+y^2+z^2=a^2$ and whose generators are parallel to the line $x/l=y/m=z/n$ is given by the equation

$$(lx+my+nz)^2=(l^2+m^2+n^2)\,(x^2+y^2+z^2-a^2).$$

(*Gorakhpur*, 1972; *Agra*, 1973; *Kanpur*, 1976)

Imp

6. Show that the equation of the cylinder with generators parallel to z-axis and whose guiding curve is

$$ax^2+by^2=2z,\quad lx+my+nz=p$$

Put z   Find z

is given by the equation

$$n\,(ax^2+by^2)+2\,(lx+my)-2p=0$$

Hint. Eliminate z between the given equations.

Unit-2

17.4 The Cone. *Definition.* A surface generated by a straight line passing through a fixed point and intersecting a given curve is called a cone.

The fixed point is called the **vertex** of the cone.

If the moving straight line through the vertex makes a constant angle with a fixed straight line also passing through the vertex the surface thus generated is called a **right circular cone.** In this case

the fixed straight line is called the *axis* of the cone and the fixed angle its *semi-vertical angle*.

17.4 Equation of a cone. *We shall now find the equation of the cone whose vertex is the point* (α, β, γ) *and base the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0.$$

The equation to a straight line passing through (α, β, γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{...(1)}$$

This meets the plane $z=0$ in the point

$$\left(\alpha-\frac{l\gamma}{n},\ \beta-\frac{m\gamma}{n},\ 0\right).$$

The above point lies on the given conic, if (see § 17.2)

$$a(n\alpha-l\gamma)^2+2h(n\alpha-l\gamma)(n\beta-m\gamma)+b(n\beta-m\gamma)^2$$
$$+2gn(n\alpha-l\gamma)+2fn(n\beta-m\gamma)+cn^2=0. \qquad \text{...(2)}$$

Now, in order to find the equation of the surface generated by (1) when it always intersects the given conic, we eliminate l, m, n between (1) and (2). We then get

$$a(z\alpha-x\gamma)^2+2h(z\alpha-x\gamma)(z\beta-y\gamma)+b(z\beta-y\gamma)^2$$
$$+2g(z-\gamma)(z\alpha-x\gamma)+2f(z-\gamma)(z\beta-y\gamma)+c(z-\gamma)^2=0$$

which is the required *equation* of the cone.

17.6 Equation of the cone when the vertex is at the origin.

Let the origin O be the vertex of the cone and let P be any point (α, β, γ) on the cone. Since OP lies wholly on the cone, the coordinates $(\lambda\alpha, \lambda\beta, \lambda\gamma)$, which represent a point of OP for arbitrary values λ, satisfy the equation of the cone for all values of λ, which, therefore, must be homogeneous.

Corollary. If the homogeneous equation $f(x, y, z)=0$ represents a cone, one of whose generators is $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$, then $f(l, m, n)=0$.

17.61 Condition for the general equation of second degree to represent a cone and to find its vertex.

notes:

A general equation of second degree in x, y, z is

$$ax^2+by^2+cz^2+2fyz+2gxz+2hxy+2ux+2vy+2wz+d=0 \qquad \text{...(1)}$$

If it represents a cone with vertex at (α, β, γ), then transferring the origin to the vertex, equation (1) becomes

$$a(X+\alpha)^2+b(Y+\beta)^2+c(Z+\gamma)^2+2f(Y+\beta)(Z+\gamma)+2g(Z+\gamma)(X+\alpha)+2h(X+\alpha)(Y+\beta)+2u(X+\alpha)+2v(Y+\beta)+2w(Z+\gamma)+d=0$$

or
$$\left.\begin{aligned}&aX^2+bY^2+cZ^2+2fYZ+2gXZ+2hXY\\&+2X(a\alpha+h\beta+g\gamma+u)+2Y(h\alpha+b\beta+f\gamma+v)\\&+2Z(g\alpha+f\beta+c\gamma+w)+a\alpha^2+b\beta^2\\&+c\gamma^2+2g\gamma\alpha+2f\beta\gamma+2h\alpha\beta+2u\alpha+2v\beta\\&+2w\gamma+d=0.\end{aligned}\right\} \text{...(2)}$$

Since this represents a cone with vertex at the origin, the coefficients of X, Y, Z and constant term in (2) must be zero separately. Therefore

$$a\alpha+h\beta+g\gamma+u=0, \qquad \text{...(3)}$$
$$h\alpha+b\beta+f\gamma+v=0, \qquad \text{...(4)}$$
$$g\alpha+f\beta+c\gamma+w=0, \qquad \text{...(5)}$$

and
$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta+2u\alpha+2v\beta+2w\gamma+d=0. \qquad \text{...(6)}$$

We can write (6) as

$$\alpha(a\alpha+h\beta+g\gamma+u)+\beta(h\alpha+b\beta+f\gamma+v)+\gamma(g\alpha+f\beta+c\gamma+w)+(u\alpha+v\beta+w\gamma+d)=0.$$

By virtue of (3), (4) and (5), this becomes

$$u\alpha+v\beta+w\gamma+d=0. \qquad \text{...(7)}$$

Eliminating α, β, γ between (3), (4), (5) and (7), the required condition for the equation to represent a conic is

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}=0. \qquad \text{...(8)}$$

When this condition is satisfied, the coordinates of vertex may be obtained by solving any three equations (3), (4), (5) and (7).

Working Rule. Denoting the general equation of second degree (1) by $F(x, y, z)$, we introduce a new variable t in this equation, so that it becomes a homogeneous equation of second degree in x, y, z and t. Thus

$$F(x, y, z, t)\equiv ax^2+by^2+cz^2+2fyz+2gxz+2hxy+2uxt+2vyt+2wzt+dt^2=0. \qquad \text{...(9)}$$

Differentiating partially with respect to x, y, z and t,

$$\frac{\partial F}{\partial x}=2(ax+hy+hz+ut)=0 \qquad \text{...(3)}$$

$$\frac{\partial F}{\partial y}=2\,(hx+by+fz+vt)=0 \qquad \ldots(4')$$

$$\frac{\partial F}{\partial z}=2\,(gx+fy+cz+wt)=0 \qquad \ldots(5')$$

$$\frac{\partial F}{\partial t}=2\,(ux+vy+wz+d)=0. \qquad \ldots(6')$$

Writing $t=1$ in the above, we obtain equations similar to (3), (4), (5) and (7). Solving (3′), (4′), (5′) and (6′) for x, y, z, we get the coordinates of vertex of the cone.

Example. Prove that the equation

$$ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$$

represents a cone if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$.

(*Meerut*, 1969; *Rajasthan*, 1975)

Introducing a variable t to make the given equation homogeneous of second degree in x, y, z, t, we obtain

$$F(x, y, z, t)\equiv ax^2+by^2+cz^2+2uxt+2vyt+2wzt+dt^2=0.$$

Consequently,

$$\frac{\partial F}{\partial x}=2\,(ax+ut)=0,\quad \frac{\partial F}{\partial y}=2\,(by+vt)=0$$

$$\frac{\partial F}{\partial z}=2\,(cz+wt)=0 \text{ and } \frac{\partial F}{\partial t}=2\,(ux+vy+wz+dt)=0$$

Putting $t=1$, in the above,

$$x=-\frac{u}{a},\ y=\frac{-v}{b},\ z=\frac{-w}{c} \text{ and } ux+vy+wz+d=0$$

or

$$u\left(\frac{-u}{a}\right)+v\left(\frac{-v}{b}\right)+w\left(\frac{-w}{c}\right)+d=0$$

or

$$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d.$$

Examples

1. Show that the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ where $l^2-2m^2+3n^2=0$ generates the cone $x^2-2y^2+3z^2=0$.

2. Show that the lines through the points (1, 2, 3) whose direction ratios satisfy the relation $2l^2+3m^2-4n^2=0$ generate the cone

$$2x^2+3y^2-4z^2-4x-12y+24z-22=0.$$

3. Find the equation to the cone whose vertex is (α, β, γ) and base is $y^2=4ax,\ z=0$.

(*Gorakhpur*, 1971)

Ans. $z^2(\beta^2-4a\alpha)-2z\gamma\,[\beta y-2a\,(x+\alpha)]+\gamma^2\,(y^2-4ax)=0.$

4. Find the equation of the right circular cone whose vertex is (α, β, γ), semi-vertical angle α and axis the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

(*Lucknow*, 1979)

Solution. Let (x_1, y_1, z_1) be any point on the cone. The direction cosines of the line joining (x_1, y_1, z_1) and (α, β, γ) are proportional to $x_1-\alpha,\ y_1-\beta,\ z_1-\gamma$. Consequently

$$\frac{(x_1-\alpha)\,l+(y_1-\beta)\,m+(z_1-\gamma)\,n}{\sqrt{[(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2]}\,\sqrt{(l^2+m^2+n^2)}}=\cos\alpha.$$

Equation of the cone is therefore

$$[l\,(x-\alpha)+m\,(y-\beta)+n\,(z-\gamma)]^2$$
$$=(l^2+m^2+n^2)\,[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]\cos^2\alpha.$$

5. Find the equation of the right circular cone which passes through (1, 1, 2) and has its vertex at origin and axis the line

$$x/2=y/(-4)=z/3.$$

Ans. $4x^2+40y^2+19z^2-72yz+36zx-48xy=0.$

6. Prove that the general equation of the cone of second degree which passes through the axes is

$$fyz+gxz+hxy=0.$$

Hint. The homogeneous equation of second degree in x, y, z must be satisfied by the direction cosines of the axis, i.e., 1, 0, 0; 0, 1, 0 and 0, 0, 1.

7. Find the equation of the cone which has vertex at the origin and passes through the curve → plane. Homogenise, surface by plane.

$$ax^2+by^2=2z,\quad lx+my+nz=p.$$

(*Gorakhpur*, 1976)

3D surface *Ans.* $p\,(ax^2+by^2)-2z\,(lx+my+nz)=0.$

8. Find the equation of the cone with vertex at origin and base the circle $x=a,\ y^2+z^2=b^2$. Show that the section of this cone by a plane parallel to the plane XOY is a hyperbola. homogenise

(*Meerut*, 1973)

Ans. $a^2\,(y^2+z^2)-b^2x^2=0.$

notes

9. Find the equation of a cone whose vertex P is the point (α, β, γ) and whose guiding curve is the conic $x^2/a^2+y^2/b^2=1$. If the section of this cone by the plane $x=0$ be a rectangular hyperbola, find the locus of P.

(*U. P. C. S.*, 1974; *Gorakhpur*, 1976)

Ans. $x^2/a^2+y^2/b^2+z^2/c^2=1.$

10. Prove that the equation

$$2x^2+2y^2+7z^2-10yz-10zx+2x+2y+26z-17=0$$

represents a cone with its vertex at (2, 2, 1).

11. Prove that the equation of a right circular cone with vertex at (2, 1, −3), axis parallel to y-axis and semi-vertical angle 45° is

$$x^2+y^2+z^2-4x+2y+6z+12=0. \qquad \textit{(Meerut, 1969)}$$

12. Lines are drawn from the origin O with direction cosines proportional to 1, 2, 2; 2, 3, 6; 3, 4, 12. Prove that the axis of the right circular cone hrough them has direction cosines proportional to −1, 1, 1, and that the semi-vertical angle of the cone is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. *(Meerut, 1973)*

17.7 Tangent plane to a cone. We shall now obtain the equation of the tangent plane at a point (α, β, γ) of the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

Since (α, β, γ) lies on the cone, we have

$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0. \qquad ...(1)$$

Now, the equation to a line passing through (α, β, γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \qquad ...(2)$$

(l, m, n) being the direction cosines of the line. Any point on (2) is given by

$$x=\alpha+lr,\ y=\beta+mr,\ z=\gamma+nr.$$

If this point lies on the given cone, we have

$$a(\alpha+lr)^2+b(\beta+mr)^2+c(\gamma+nr)^2 +2f(\beta+mr)(\gamma+nr)+2g(\gamma+nr)(\alpha+lr) +2h(\alpha+lr)(\beta+mr)=0,$$

or

$$r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hlm) +2r\{l(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+f\gamma) +n(g\alpha+f\beta+c\gamma)\}+a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma +2g\gamma\alpha+2h\alpha\beta=0. \qquad ...(3)$$

One root of equation (3) is equal to zero by virtue of relation (1). If the line (2) is a tangent to the cone, the other root of equation (3) must also be equal to zero, the condition for which is

$$l(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+f\gamma)+n(g\alpha+f\beta+c\gamma)=0. \qquad ...(4)$$

Elliminating l, m, n between (2), (3) and (4), the equation of the tangent plane at (α, β, γ) is

$$(x-\alpha)(a\alpha+h\beta+g\gamma)+(y-\beta)(h\alpha+b\beta+f\gamma) +(z-\gamma)(g\alpha+f\beta+c\gamma)=0,$$

which, with the help of relation (1), can be written as

$$x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma) +z(g\alpha+f\beta+c\gamma)=0. \qquad ...(5)$$

17.71 A property of the tangent plane. We shall now prove that *the tangent plane touches the cone at every point of a generator.*

We see from equation (5) that the tangent plane at (α, β, γ) passes through the vertex (0, 0, 0). From § 17.6 coordinates of any point on the generator through (α, β, γ) are $(\lambda\alpha, \lambda\beta, \lambda\gamma)$. Now, from (5), we see that the tangent plane at (α, β, γ) is also the tangent plane at $(\lambda\alpha, \lambda\beta, \lambda\gamma)$. This proves the proposition.

17.72 Condition of tangency of the plane ux+vy+wz=0.

From § 17.7, the tangent plane at a point (α, β, γ) of the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

is

$$x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0.$$

The plane $ux+vy+wz=0$ therefore touches the cone along the generator through (α, β, γ) if

$$\frac{u}{a\alpha+h\beta+g\gamma}=\frac{v}{g\alpha+b\beta+f\gamma}=\frac{w}{g\alpha+f\beta+c\gamma}. = \frac{1}{\lambda}\text{ (say)}$$

Putting each of these equal to $\frac{1}{\lambda}$, we get

$$a\alpha+h\beta+g\gamma-u\lambda=0, \qquad ...(1)$$

$$h\alpha+b\beta+f\gamma-v\lambda=0, \qquad ...(2)$$

$$g\alpha+f\beta+c\gamma-w\lambda=0. \qquad ...(3)$$

Also, since (α, β, γ) lies on the given plane,

$$u\alpha+v\beta+w\gamma=0. \qquad ...(4)$$

Eliminating α, β, γ and λ from (1), (2), (3) and (4), we get

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}=0. \qquad ...(5)$$

as *condition of tangency.*

17.8 Reciprocal cone. The condition of tangency [equation (5) of the preceding article] can be written as

$$Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv=0.$$

The normal $\frac{x}{u}=\frac{y}{v}=\frac{z}{w}$ through the vertex, i.e., the origin, to the plane $ux+vy+wz=0$, therefore, generates the cone

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0 \quad ...(1)$$

where $A=bc-f^2$, $B=ca-g^2$, $C=ab-h^2$, $F=gh-af$, $G=hf-bg$, $H=fg-ch$.

Now it is easy to verify that a normal through the origin to a tangent plane to cone (1) generates the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0. \quad ...(2)$$

Cone (1) and (2), which are such that each is the locus of the normals drawn through the vertex to the tangent planes of the other, are called **reciprocal cones.**

17.9 Cone with three mutually perpendicular generators. We shall now obtain the condition which must be satisfied in order that the cone $ax^2+by^2-cz^2+2fyz+2gzx+2hxy=0$ may have three mutually perpendicular generators. For this, we shall find the condition that the plane $ux+vy+wz=0$ should cut the cone in two perpendicular generators.

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be one of the lines of intersection of the plane and the cone. Then

$$ul+vm+wn=0 \quad ...(1)$$

and

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0. \quad ...(2)$$

Eliminating n between (1) and (2), we get

$$l^2(cu^2+aw^2-2gwu)+2lm\ (hw^2-cuv+fuw-gvw)+m^2\ (cv^2+bw^2-2fvw)=0.$$

If $\frac{l_1}{m_1}$, $\frac{l^2}{m^2}$ be the roots of the above equation, we have

$$\frac{l_1l_2}{m_1m_2}=\frac{cv^2+bw^2-2fw}{cu^2+aw^2-2gwu}.$$

We obtain, similarly,

$$\frac{l_1l_2}{n_1n_2}=\frac{cv^2+bw^2-2fvw}{bu^2+av^2+2huv}$$

From (3) and (4),

$$\frac{l_1l_2}{cv^2+bw^2-2fvw}=\frac{m_1m_2}{cu^2+aw^2-2gwu}=\frac{n_1n_2}{bu^2+av^2-2huv}.$$

The lines of intersection of the plane and the cone are therefore perpendicular if

$$u^2(b+c)+v^2(c+a)+w^2(a+b)-2fvw-2gwu-2huv=0. \quad ...(5)$$

Further, if the normal to the plane also lies on the cone, we have

$$au^2+bv^2+cw^2+2fvw+2gwu+2huv=0. \quad ...(6)$$

Adding (5) and (6) and removing the common factor $u^2+v^2+w^2$, we get

$$a+b+c=0$$

as the condition that the cone should have three mutually perpendicular generators.

17.91 Number of mutually perpendicular generators.

We shall now show that if $a+b+c=0$, the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ has an infinity of sets of mutually perpendicular generators.

Let the normal to the plane $ux+vy+wz=0$ lie on the given cone. Then,

$$au^2+bv^2+cw^2+2fvw+2gw+2huv=0 \quad ...(1)$$

Also, we are given that

$$a+b+c=0. \quad ...(2)$$

Multiplying (2) by $(u^2+v^2+w^2)$ and substracting (1) from the resulting equation, we get equation (5) of the preceding article. The given plane thus cuts the cone in two perpendicular generators.

We thus see that any plane through the origin which is normal to a generator of the cone cuts in two perpendicular generators if $a+b+c=0$, and the cone thus has an infinite number of sets of mutually perpendicular generators.

Example. Show that the cone $3yz-2zx-2xy=0$ has an infinite set of three mutually perpendicular generators. If $\frac{x}{1}=\frac{y}{1}=\frac{z}{2}$ be a generator belonging to one such set, find the other two. (*Ranchi*, 1968)

Solution. Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be one of the other two generators. Then we have

$$l+m+2n=0, \quad ...(1)$$

and

$$3mn-2nl-2lm=0. \quad ...(2)$$

Substituting the value of l obtained from (1) in (2), we have on simplification

$$2m^2+9mn+4n^2=0. \quad ...(3)$$

Solving (3),

$$m=-\frac{n}{2} \text{ or } m=-4n$$

Hence from (1),

$$l=-\frac{3n}{2} \text{ or } l=2n.$$

The other two generators are, therefore,

$$\frac{x}{3}=\frac{y}{1}=\frac{z}{-2}$$

and $$\frac{x}{2}=\frac{y}{-4}=\frac{z}{1}.$$

Examples

1. Find the equation to the lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$. Also find the angle between the lines of section.

(*Rajasthan*, 1971; *Rohilkhand*, 1977)

Ans. $\frac{x}{-1}=\frac{y}{4}=\frac{z}{2}$; $\frac{x}{-1}=\frac{y}{2}=\frac{z}{0}$; $\cos^{-1}\left(\frac{9}{\sqrt{105}}\right)$

2. Prove that the angle between the lines given by $x+y+z=0$, $ayz+bzx+cxy=0$ is $\frac{1}{2}\pi$ when $a+b+c=0$ and $\frac{1}{3}\pi$ when $1/a+1/b+1/c=0$. (*Meerut*, 1975)

3. If the plane $2x-y+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines, find the value of c. (*Agra*, 1972)

Ans. $c=z$.

4. Find the condition that the lines of the section of the plane $lx+my+nz=0$ and cones $fyz+gzx+hxy=0$, $ax^2+by^2+cz^2=0$ be coincident. (*U. P. C. S.*, 1972; *Meerut*, 1977)

Ans. $\frac{fmn}{bn^2+cm^2}=\frac{gnl}{cl^2+an^2}=\frac{hlm}{am^2+bl^2}$

5. Show that the cones $ax^2+by^2+cz^2=0$ and $x^2/a+y^2/b+z^2/c=0$ are reciprocal.

(*Lucknow*, 1965; *Jodhpur*, 1967; *Kanpur*, 1974)

6. Prove that perpendiculars drawn from the origin to tangent planes to the cone

$$3x^2+4y^2+5z^2+2yz+4zx+6xy=0$$

lie on the cone

$$19x^2+11y^2+3z^2+6yz-10zx+26xy=0.$$

(*Lucknow*, 1978)

7. Prove that the plane $lx+my+nz=0$ cuts the cone $(b-c)x^2+(c-a)\ y^2+(a-b)\ z^2-2fyz+2gzx+2hxy=0$ in perpendicular lines if

$$(b-c)\ l^2+(c-a)\ m^2+(a-b)\ m^2+2fmn+2gnl+2hlm=0,$$

Examples on Chapter XVII

1. Find the equation of the circular cylinder whose guiding curve is the circle

$$x^2+y^2+z^2=9,\ x-y+z=3.$$

(*Kanpur*, 1970; *Lucknow*, 1972)

Ans. $x^2+y^2+z^2+xy+yz-zx-9=0.$

2. Show that the line $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$, where $2l^2+3m^2-4n^2=0$ is a generator of the cone

$$2(x-\alpha)^2+3(y-\beta^2)-4(z-\gamma)^2=0.$$

3. Find the condition that the plane $lx+my+nz=0$ may touch the cone $ax^2+by^2+cz^2=0$.

Ans. $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0;$

4. Show that the plane $ax+by+cz=0$ cuts the cone $xy+yz+zx=0$ in perpendicular lines if $a^{-1}+b^{-1}+c^{-1}=0$.

(*Lucknow*, 1980; *Rajasthan*, 1976)

5. Show that the lines given by $x-y-z=0$, $ayz+bzx+cxy=0$ are at right angles if $a=b+c$.

6. Find the equation to the cone whose vertex is the origin and which passes through the curve of intersection of the plane $lx+my+nz=p$ and the surface $ax^2+by^2+cz^2=1$.

Ans. $p^2(ax^2+by^2+cz^2)=(lx+my+nz)^2$.

7. Prove that the equation of the cone whose vertex is (x', y', z') and base the curve $x^2/a^2+y^2/b^2=1$, $\cancel{z}=0$ is $z=0$

$$(z-\cancel{z}')^2=\frac{(xz'-x'z)^2}{a^2}+\frac{(yz'-y'z)^2}{b^2}.$$

8. Show that the general equation of a cone which touches the coordinate planes is

$$a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy=0.$$ → on notes.

(*Lucknow*, 1970; *Meerut* 1970; *Agra*, 1975)

9. Prove that the equation gy

$$\sqrt{fx}+\sqrt{gx}+\sqrt{hz}=0$$

respects a cone which touches the coordinate planes, and that the equation of the reciprocal cone is

$$fyz+gxz+hxy=0.$$

(*Kanpur*, 1975; *Agra*, 1977; *I. A. S.*, 1976)

10. Find the equation of the right circular cone whose semi-vertical angle is α, the axis is the axis of z and vertex the origin.

(*Agra*, 1975; *Lucknow*,, 1978)

Ans. $x^2+y^2=z^2 \tan^2\alpha$.

11. Find the vertical angle of a right circular cone having three mutually perpendicular generators.

Ans. $2 \tan^{-1} \sqrt{2}$.

12. Planes through OX and OY include an angle α, Show that their line of intersection lies on the cone

$$z^2 (x^2+y^2+z^2)=x^2y^2 \tan^2 \alpha.$$

(*Meerut*, 1976; *Lucknow*, 1980)

13. Show that the condition that the cone

$$az^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

has three mutually perpendicular generators if

$$bc+ca+ab=f^2+g^2+h^2$$

(*Delhi*, 1971; *Kanpur*, 1977)

14 Find the equation of the cone generated by rotating the line $x/l=y/m=z/n$ about the line $x/a=b/y=z/c$.

Ans. $(al+bm+cn)(x^2+y^2+z^2)=(l^2+m^2+n^2)(ax+by+cz)^2$.

15. Find the plane which touches the cone

$$2x^2+y^2-5z^2+3yz-6zx+5xy=0$$

along the generator whose direction ratios are 1, 1, 1.

Ans. $3x+10y-13z=0$.

Same as ex 365

16. If $x/1=y/2=z/3$ represent one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the equation of the other two.

(*Meerut*, 1976; (*Lucknow*, 1978)

Ans. $\boxed{x=y=-z}$; $4x=-5y=20z$.

17. Shaw that semi-vertical angle of a right circular cone which has three mutually perpendicular tangent planes is $\cot^{-1} \sqrt{2}$.

18. Prove that the plane which cut $ax^2+by^2+cz^2=0$ in perpendicular generators touch the cone

$$\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0. \quad (\textit{Kanpur}, 1976)$$

Hint. The plane $ux+vy+wz=0$ cuts the conic in perpendicular generators if $(b+c)u^2+(c+a)v^2+(a+b)w^2=0$. The normal to the plane generators the cone

$$(b+c)x^2+(c+a)y^2+(a+b)z^2=0.$$

Find the reciprocal cone.

19. Show that the equation to the cone whose vertex is the origin and base the curve $z=k, f(x, y)=0$ is

$$f\left(\frac{xk}{z}, \frac{yk}{z}\right)=0.$$

20. Find the equation of the cone which has its vertex at $(2a, b, c)$ and passes through the curve $x^2+y^2+=4a^2$, $z=0$. Find b and c if the cone also passes through the curve $y^2=4a\ (z+a)$, $x=0$. Show that the plane $y=0$ cuts the cone in two straight lines. Find the angle between them.

Ans. $b=0, c=-2a; \cos^{-1} \frac{1}{\sqrt{5}}$.

21. Find the angle between the lines given by $x+y+y=0$ and the cone

$$\frac{yz}{q\ r}+\frac{zx}{r-p}+\frac{xy}{p-q}=0.$$

Ans. $\frac{1}{3}\pi$.

22. Prove that the locus of the line of intersection of tangent planes to the cone $ax^2+by^2+cz^2=0$ which touch along perpendicular generators is the cone

$$a^2 (b+c) x^2+b^2 (c+a) y^2+c^2 (a+b) z^2=0.$$

(*Kanpur*, 1977; *U. P. C. S.*, 1969)

Scanned by CamScanner

CHAPTER XVIII

Unit-3

THE CENTRAL CONICOIDS

18.1 The equation $ax^2+by^2+cz^2=1$ represents a *central conicoid*, the centre being at the origin. This is a particular case of the general equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0.$$

18.2 The tangent plane. We shall now obtain the equation of the tangent plane at a point (α, β, γ) of the central conicoid $ax^2+by^2+cz^2=1$.

Let the straight line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r, \quad \ldots(1)$$

where l, m, n are direction cosines of the line and r the distance of (x, y, z) from (α, β, γ), meet the conicoid in two points P and Q. The coordinates of P (or Q) can be written as

$$(\alpha+lr, \beta+mr, \gamma+nr)$$

where r is now the distance of P (or O) from (α, β, γ).

Substituting the above coordinates of the equation of the conicoid and arranging terms in powers of r, we get

$$r^2 (al^2+bm^2+cn^2)+2r (a\alpha l+b\beta m+c\gamma n)$$
$$+a\alpha^2+b\beta^2+c\gamma^2-1=0. \quad \ldots(2)$$

Equation (2) gives the distances of P and Q from the given point (α, β, γ). If $a\alpha^2+b\beta^2+c\gamma^2-1=0$, the point (α, β, γ) lies on the conicoid. Further, if

$$a\alpha l+b\beta m+c\gamma n=0. \quad \ldots(3)$$

18.4

the line (1) is a tangent to the conicoid at (α, β, γ).

Eliminating (l, m, n) between (1) and (3), we obtain

$$(x-\alpha)\, a\alpha+(y-\beta)\, b\beta+(z-\gamma)\, c\gamma=0,$$

18.3

or

$$a\alpha x+b\beta y+c\gamma z=1$$

as the equation to the locus of all the tangent lines through (α, β, γ). This is a plane.

The equation of the **tangent plane** at the point (α, β, γ) is thus

$$\boxed{a\alpha x+b\beta y+c\gamma z=1.}$$

18.3 Condition of tangency of the plane $lx+my+nz=p$.

We shall now find the condition that the plane $lx+my+nz=p$ should touch the conicoid $ax^2+by^2+cz^2=1$.

We know that the equation of the tangent plane at (α, β, γ) is

$$a\alpha x+b\beta y+c\gamma z=1. \quad \ldots(1)$$

For same value of (α, β, γ) this will be the same as the equation of the given plane. Comparing coefficients,

$$\frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{c\gamma}{n}=\frac{1}{p}$$

i.e.,

$$\alpha=\frac{l}{ap}, \beta=\frac{m}{bp}, \gamma=\frac{n}{cp}$$

Since (α, β, γ) lies on the conicoid,

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2.$$

This is the required *condition of tangency*.

18.3 Section with a given centre. If (α, β, γ) be the mid-point of the chord

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

the roots of equation (2) of § 18.2 must be equal in magnitude and opposite in sign. The condition for this is

$$a\alpha l+b\beta m+c\gamma n=0.$$

Eliminating l, m, n which the help of the equation of the chord, we get

$$(x-\alpha)\, \cancel{ax}\,a\alpha+(y-\beta)\, b\beta+(z-\gamma)\, c\gamma=0$$

as the equation of section whose centre is the point (α, β, γ).

18.4 Locus of mid-points of a system of parallel chords.

Let chords of the conicoid $ax^2+by^2+cz^2=1$ be drawn parallel to the fixed line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

If one such chord be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

where (α, β, γ) is the mid-point of the chord, we have from the preceding article,

$$al\alpha+bm\beta+cn\gamma=0.$$

The required locus is therefore

$$alx + bmy + cnz = 0.$$

This is called the *diametral plane* bisecting the given system of parallel chords.

18.5 The polar plane. *Definition.* If any secant, APQ, through a given point A meets a conicoid in P and Q, and R is the harmonic conjugate of A with respect to P and Q, the locus of R is called the polar of A with respect to the conicoid.

Let A be the point (α, β, γ) and let the equation to APQ be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

where l, m, n, are the direction cosines of APQ.

If now, r_1 and r_2 denote the lengths AP and AQ, they are the roots of the equation

$$r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)$$
$$+(a\alpha^2+b\beta^2+c\gamma^2-1)=0.$$

If the length AR be r',

$$r'=\frac{2r_1r_2}{r_1+r_2}=-\frac{a\alpha^2+b\beta^2+c\gamma^2-1}{a\alpha l+b\beta m+c\gamma n}$$

Let (x', y', z') be the coordinates of R. Then from the equations to the line,

$$x'-\alpha=lr',\ y'-\beta=mr',\ z'-\gamma=nr'.$$

From the above we now get

$$(x'-\alpha)\,a\alpha+(y'-\beta)\,b\beta+(z'-\gamma)\,c\gamma$$
$$=-(a\alpha^2+b\beta^2+c\gamma^2-1).$$

Hence the locus of R is

$$a\alpha x+b\beta y+c\gamma z=1.$$

This is the **polar plane** of (α, β, γ).

18.51 Polar lines. From the equation of the polar plane of a point it is evident that if the polar plane of (x_1, y_1, z_1) passes through (x_2, y_2, z_2), then the polar plane of (x_2, y_2, z_2) passes through (x_1, y_1, z_1). Hence if the polar plane of any point on a line PQ passes through a line $P'Q'$, then the polar plane of any point $P'Q'$ passes through that point on PQ, and therefore, passes through PQ. The lines PQ and $P'Q'$ are then said to be **polar lines** with respect to the given conicoid.

To find the polar line of the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{...(1)}$$

with respect to the conicoid

$$ax^2+by^2+cz^2=1, \quad \text{...(2)}$$

we proceed as follows :

The polar plane of any point $(\alpha+lr, \beta+mr, \gamma+nr)$ on (1) with respect to (2) is

$$a\alpha x+b\beta y+c\gamma z-1+r(alx+bmy+cnz)=0.$$

This obviously passes through the line

$$a\alpha x+b\beta y+c\gamma z-1=0,\ alx+bmy+cnz=0 \quad \text{...(3)}$$

for all values of r.

The line (3) is therefore the polar line of (1).

18.6 The enveloping cone. If the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

passes through A, (α, β, γ) meets the conicoid $ax^2+by^2+cz^2=1$ in P and Q, from Art. 18.2, the measures of AP and AQ are the roots of the quadratic

$$r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)$$
$$+(a\alpha^2+b\beta^2+c\gamma-1)=0.$$

If P and Q coincide, the line becomes a tangent. The condition for this is

$$(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1)$$
$$=(a\alpha l+b\beta m+c\gamma n)^2.$$

The line thus lies on the cone

$$[a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2][a\alpha^2+b\beta^2+c\gamma^2-1]$$
$$=[a\alpha(x-\alpha)+b\beta(y-\beta)+c\gamma(z-\gamma)]^2.$$

This can be written as

$$(ax^2+by^2+cz^2-1)(a\alpha^2+b\beta^2+c\gamma^2-1)$$
$$=(a\alpha x+b\beta y+c\gamma z-1)^2,$$

or as $\quad SS_1=P^2$

where $\quad S\equiv ax^2+by^2+cz^2-1,$

$\quad S_1\equiv a\alpha^2+b\beta^2+c\gamma^2-1,$

and $\quad P\equiv a\alpha x+b\beta y+c\gamma z-1.$

The above cone is called the **enveloping cone** of the conicoid.

Examples

1. Find the point of contact of the plane $4x-6y+3z=5$ and the conicoid

$$2x^2-6y^2+3z^2=5.$$

Ans. (2, 1, 1).

2. Find the equation of the tangent plane to the conicoid $x^2+2y^2+z^2=4$, which passes through the line

$$x+y+z+1=0=2x+3y+2z-3.$$

Solution. The equation of the plane passing through the given line is

$$x+y+z+1+\lambda\,(2x+3y+2z-3)=0$$

or $$x\,(1+2\lambda)+y\,(1+3\lambda)+z\,(1+2\lambda)\;1-3\lambda=0. \qquad ...(1)$$

This touches the given conicoid if

$$\frac{(1+2\lambda)^2}{1/4}+\frac{(1+3\lambda)^2}{2/4}+\frac{(1+2\lambda)^2}{1/4}=(3\lambda-1)^2,$$

i.e., $41\lambda^2+50\lambda+9=0$ giving $\lambda=-1$ or $(-9/41)$.

Substituting these values of λ in (1), the required tangent planes are

$$x+2y+z=4, \qquad 23x+14y+23z+68=0.$$

3. Find the equation of the planes to $7x^2+5y^2+3z^2=60$ which pass through the line $7x+10y=30$, $5y-3z=0$ (*Kanpur*, 1974)

On नोट

Ans. $7x+5y+3z=30$; $14x+10y+9z=50$.

4. Find the equation of tangent planes to $2x^2-6y^2+3z^2=5$ which pass through the line

$$x+9y-3z=0 \text{ and } 3x-3y+6z=5.$$

(*Kurukshetra*, 1973)

Ans. $4x+6y+3z=5$ and $2x-12y+9z=5$.

5. A tangent plane to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

meets the coordinate planes in P, Q, R. Find the locus of the centroid of the triangle PQR. (*Agra*, 1975; *Kanpur*, 1977)

Ans. $a^2/x^2+b^2/y^2+c^2/z^2=9$.

6. Find the locus of the foot of the perpendicular f om the origin on varying tangent planes to the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

(*Rajasthan*, 1975; *Kanpur*, 1975)

Ans. $(x^2+y^2+z^2)^2=a^2x^2+b^2y^2+c^2z^2$.

3, 5, 6, 8, 15, 13, 17

H/A: 12, 4, 7, 9, 10, 14, 16

H.W

7. Find the equation to two tangent planes to the surface $ax^2+by^2+cz^2=1$ which pass through the line

$$u\equiv lx+my+nz-p=0, \quad v\equiv l'x+m'y+n'z-p'=0$$

Ans. $$u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2uv\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)$$
$$+v^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0.$$

8. Find the equation to the plane which cuts the surface $2x^2-3y^2+5z^2=1$ in a conic whose centre is at (2, 1, 3).

αβγ (*Lucknow*, 1978)

Ans. $4x-3y+15z=50$.

9. Find the locus of the middle points of chords of the surface $ax^2+by^2+cz^2=1$ which pass through (α, β, γ).

Ans. $ax\,(x-\alpha)+by\,(y-\beta)+cz\,(z-\gamma)=0$.

10. Find the locus of the centres of the section of the surface $ax^2+by^2+cz^2=1$ which touch $\alpha x^2+\beta y^2+\gamma z^2=1$.

Solution. Let (f, g, h) be the centre of a section of the first surface. The equation of this section is

$$af\,(x-f)+bg\,(y-g)+ch\,(z-h)=0,$$

or $$afx+bgy+chz=af^2+bg^2+ch^2.$$

This plane will touch the second surface if

$$\frac{a^2f^2}{\alpha}+\frac{b^2g^2}{\beta}+\frac{c^2h^2}{\gamma}=(af^2+bg^2+ch^2)^2.$$

Consequently, the locus of (f, g, h) is

$$\frac{a^2x^2}{\alpha}+\frac{b^2y^2}{\beta}+\frac{c^2z^2}{\gamma}=(ax^2+by^2+cz^2)^2.$$

11. Find the equation of the polar of the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

with respect to the conicoid $ax^2+by^2+cz^2=1$.

Solution. Any point on the given line is $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$.

The polar plane of this point is

$$a\alpha x+b\beta y+c\gamma z-1+r\,(alx+bmy+cnz)=0.$$

For all values of r, this passes through the line

$$a\alpha x+b\beta y+c\gamma z-1=0=alx+bmy+cnz.$$

This line is called the **polar** of the given line.

12. Prove that the locus of the poles of the tangent planes of $a^2x^2+b^2y^2+c^2z^2=1$ with respect to $\alpha^2x^2+\beta^2y^2+\gamma^2z^2=1$ is

$$\frac{\alpha^4x^2}{a^2}+\frac{\beta^4y^2}{b^2}+\frac{\gamma^4z^2}{c^2}=1.$$

(*Kurukshetra*, 1974)

13. Find the locus of the pole of the plane $lx+my+nz=p$ with respect to the system of the surfaces

on notes

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1$$

where λ is a variable parameter. (*Kanpur*, 1976)

Ans. $\dfrac{(x-a^2l/p)}{l}=\dfrac{(y-b^2m/p)}{m}=\dfrac{(z-c^2n/p)}{n}.$

14. Through a fixed point (α, β, γ) straight lines are drawn at right angles to their polars with respect to $ax^2+by^2+cz^2=1$. Show that these lines lie on the cone

$$\frac{\alpha}{x-\alpha}\left(\frac{1}{b}-\frac{1}{c}\right)+\frac{\beta}{y-\beta}\left(\frac{1}{c}-\frac{1}{a}\right)+\frac{\gamma}{z-\gamma}\left(\frac{1}{a}-\frac{1}{b}\right)=0.$$

(*Meerut*, 1969; *Lucknow*, 1980)

15. Prove that the lines through (α, β, γ) at right angles to their polars with respect to the surface

$$\frac{x^2}{a+b}+\frac{y^2}{2a}+\frac{z^2}{2b}=1$$

generate the cone

$$(y-\beta)(\alpha z-\gamma x)+(z-\gamma)(\alpha y-\beta x)=0.$$

(*Allahabad*, 1969)

16. Find the locus of the centres of all sections of the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}$$

by all planes which pass through a fixed point (α, β, γ).

Ans. $\dfrac{x(x-\alpha)}{a^2}+\dfrac{y(y-\beta)}{b^2}+\dfrac{z(z-\gamma)}{c^2}=0.$

17. Find the equation of the tangent cone (enveloping cone) from the point $(2, -3, 6)$ on the surface $x^2+2y^2+z^2=2$.

(*I. A. S.*, 1967)

Ans. $35x^2+56y^2+19z^2+36yz-12zx+12xy$
$+4x-12y+12z=76.$

18. Find the locus of a luminous point if the surface $ax^2+by^2+cz^2=1$ casts a circular shadow on the plane $z=0$.

Ans. $x=0, aby^2+c(a-b)z^2=a-b;$
$y=0, abx^2+c(b-a)z=b-a.$

Hint. The section of the enveloping cone by the plane $z=0$ is a circle.

19. Find the locus of a luminous point which moves such that the sphere $x^2+y^2+z^2-az=0$ casts a parabolic shadow on the plane $z=0$.

(*U. P. C. S.*, 1975)

Ans. $z=a.$

20. Find the focus of the conic in which the plane $z=a$ meets any enveloping cone of the sphere $x^2+y^2+z^2=a^2$.

Ans. $(0, 0, a)$.

Unit-4

18.7 Classification of central conicoids. If we leave the sphere, we have three conicoids of which the centre is at a finite distance. These are (i) **the ellipsoid,** (ii) **the hyperboloid of one sheet,** (iii) **the hyperboloid of two sheets.** The equation of the ellipsoid referred to its principal **axes** is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad \text{...(1)}$$

that of the hyperboloid of one sheet is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \qquad \text{...(2)}$$

and that of the **hyperboloid** of two sheets is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1. \qquad \text{...(3)}$$

The section of the hyperboloid of one sheet by any plane parallel to $x=0$ or $y=0$ is a hyperbola. The section of the hyperboloid of two sheets by any plane parallel to $y=0$ or $z=0$ is also a hyperbola. It is easy to see from equations (2) and (3) that the hyperboloids are not closed surfaces. Only the ellipsoid is a closed surface. We shall now study certain loci associated with the ellipsoid.

18.8 The normal. We shall now find the equations of the normal at a point (x', y', z') of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$

The equation of the tangent plane at (x', y', z') to the ellipsoid is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1.$$

4.A → Ex. 3,5   Ex 1,3,5,6,7 / P. 378-381, Ex-2 / P. 384, H/A → Ex. 1 / P. 385

Hence the equations of the normal at (x', y', z') are

$$\frac{x-x'}{\dfrac{x'}{a^2}}=\frac{y-y'}{\dfrac{y'}{b^2}}=\frac{z-z'}{\dfrac{z'}{c^2}} \qquad ...(1)$$

The direction cosines of the normal are

$$\frac{\dfrac{x'}{a^2}}{\sqrt{\Sigma\dfrac{x'^2}{a^4}}},\quad \frac{\dfrac{y'}{b^2}}{\sqrt{\Sigma\dfrac{y'^2}{b^4}}},\quad \frac{\dfrac{z'}{c^2}}{\sqrt{\Sigma\dfrac{z'^2}{c^4}}}$$

i.e.,
$$\frac{px'}{a^2},\ \frac{py'}{b^2},\ \frac{pz'}{c^2},$$

where p is the length of the perpendicular from the origin upon the tangent plane at (x', y', z').

18.81 Normals from a given point. Suppose that normal at (x', y', z') passes through (α, β, γ). Then from equation (1) of the preceding article, we have

$$\frac{\alpha-x'}{\dfrac{x'}{a^2}}=\frac{\beta-y'}{\dfrac{y'}{b^2}}=\frac{\gamma-z'}{\dfrac{z'}{c^2}}=\lambda, \text{ say.}$$

These give

$$x'=\frac{a^2\alpha}{a^2+\lambda},\ y'=\frac{b^2\beta}{b^2+\lambda},\ z'=\frac{c^2\gamma}{c^2+\lambda}.$$

Since (x', y', z') lies on the ellipsoid,

$$\frac{a^2\alpha^2}{(a^2+\lambda)^2}+\frac{b^2\beta^2}{(b^2+\lambda)^2}+\frac{c^2\gamma^2}{(c^2+\lambda)^2}=1.$$

The above equation gives six values of λ, and therefore six normals can be drawn to the ellipsoid from any given point.

Examples

1. Show that the six normals to the ellipsoid lie on a cone.

(*Lucknow*, 1971; *Punjab*, 1972)

Solution. Let l, m, n be the direction cosines of the normal at (x', y', z') and let it pass through the fixed point (α, β, γ). Then

$$l=\frac{px'}{a^2}=\frac{p\alpha}{a^2+\lambda};\ m=\frac{b\beta}{b^2+\lambda};\ n=\frac{p\gamma}{c^2+\lambda}.$$

From these,

$$\frac{\alpha}{l}(b^2-c^2)+\frac{\beta}{m}(c^2-a^2)+\frac{\gamma}{n}(a^2-b^2)=0$$

The above equation shows that the normal lies on the cone

$$\frac{a(b^2-c^2)}{x-\alpha}+\frac{\beta(c^2-a^2)}{y-\beta}+\frac{\gamma(a^2-b^2)}{z-\gamma}=0.$$

2. The normal at a point P of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ meets the principal planes in L, M, N. Show that

$$PL:PM:PN::a^2:b^2:c^2.$$

(*Rewa*, 1970)

3. Normals are drawn from the point (α, β, γ) to the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$. Show that their feet also lie on the cone

$$\Sigma\frac{a^2(b^2-c^2)\alpha}{x}=0.$$

(*Vikram*, 1965

Hint. From § 18.81,

$$a^2+\lambda=\frac{a^2\alpha}{x'},\ b^2+\lambda=\frac{b^2\beta}{y'},\ c^2+\lambda=\frac{c^2\gamma}{z'}$$

Multiply by b^2-c^2 etc. and add.

4. Normals are drawn from P to an ellipsoid whose centre is C. Show that PC and the perpendicular from P to its polar plane lie on the same cone on which the six normals lie.

5. Prove that the lines drawn from the origin parallel to the normals to $ax^2+by^2+cz^2=1$ at its point of intersection with the plane $lx+my+nz=p$ generate the cone

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

(*Kanpur*, 1977)

Solution. Let (α, β, γ) be a point of intersection of the given plane. Then

$$a\alpha^2+b\beta^2+c\gamma^2=1 \qquad ...(1)$$

and
$$\alpha l+\beta m+\gamma n=p. \qquad ...(2)$$

The equation of the normal at (α, β, γ) to the surface is

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c\gamma}.$$

The equation of the line parallel to normal through the origin is

$$\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}. \qquad ...(3)$$

From (1) and (2), we obtain

$$p^2(a\alpha^2+b\beta^2+c\gamma^2)=(\alpha l+\beta m+\gamma n)^2 \qquad ...(4)$$

Eliminate α, β, γ, from (3) and (4).

6. If the feet of three normals from a point P to the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

lie on the plane $x/a+y/b+z/c=1$, prove that the feet of the other three lie on the plane $x/a+y/b+z/c+1=0$, and that P lies on the line

$$a(b^2-c^2)\,x=b\,(c^2-a^2)\,y=c\,(a^2-b^2)\,z.$$

Solution. Let $P:(x_1, y_1, z_1)$ the given point from where six normals are drawn to the surface. If (α, β, γ), be the coordinates of the feet of the one of the normals, then

$$\alpha=\frac{a^2x_1}{a^2+\lambda},\ \beta=\frac{b^2y_1}{b^2+\lambda},\ \gamma=\frac{c^2z_1}{c^2+\lambda}. \qquad ...(1)$$

Where the values of λ are given by equation

$$\frac{a^2x_1^2}{(a^2+\lambda)^2}+\frac{b^2y_1^2}{(b^2+\lambda)^2}+\frac{c^2z_1^2}{(c^2+\lambda)^2}-1=0. \qquad ...(2)$$

Three of the six feet of the normals lie on the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0. \qquad ...(3)$$

Therefore $\quad \dfrac{ax_1}{a^2+\lambda}+\dfrac{by_1}{b^2+\lambda}+\dfrac{cz_1}{c^2+\lambda}-1=0. \qquad ...(4)$

Let the other three feet of the normals lie on the plane

$$\frac{x}{A}+\frac{y}{B}+\frac{z}{C}-1=0. \qquad ...(5)$$

Then, similarly,

$$\frac{a^2x_1}{A\,(a^2+\lambda)}+\frac{b^2y_1}{B\,(b^2+\lambda)}+\frac{c^2z_1}{C\,(c^2+\lambda)}-1=0. \qquad ...(6)$$

The condition (2) is the product of (4) and (6). Comparing the corresponding terms in the product of (4) and (6) with (2), we obtain

$$\frac{a^2x_1^2}{A\,(a^2+\lambda)^2}=\frac{a^2x_1^2}{(a^2+\lambda)^2},$$

and similar expressions, giving

$$\frac{a}{A}=\frac{b}{B}=\frac{c}{C}=-1.$$

Putting the values of A, B, C in (5), the equation of the other plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0. \qquad ...(7)$$

This proves the first part.

From (3) and (7), the feet of the normals lie on the surface

$$\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1\right)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right)=0$$

or $\quad \left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1\right)+2\left(\dfrac{yz}{bc}+\dfrac{zx}{ca}+\dfrac{xy}{ab}\right)=0. \qquad ...(8)$

But the feet of the normals also lie on

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0.$$

Consequently, the feet of the normals also lie on the cone

$$\frac{zy}{bc}+\frac{zx}{ca}+\frac{xy}{ab}=0. \qquad ...(9)$$

But, in Example 3 above, we have seen that the feet of the normals lie on the cone

$$\frac{a^2(b^2-c^2)\,x_1}{x}+\frac{b^2(c^2-a^2)\,y_1}{y}+\frac{c^2(a^2-b^2)\,z_1}{z}=0.$$

or $\quad a^2(b^2-c^2)\,x_1yz+b^2(c^2-a^2)\,y_1zx+c^2(a^2-b^2)\,z_1xy=0. \qquad ...(10)$

Comparing the coefficients of yz, zx, xy in (9) and (10),

$$\frac{a^2(b^2-c^2)\,x_1}{1/bc}=\frac{b^2(c^2-a^2)\,y_1}{1/ca}=\frac{c^2(a^2-b^2)z_1}{1/ab}$$

or $\quad a(b^2-c^2)\,x_1=b\,(c^2-a^2)\,y_1=c\,(a^2-b^2)\,z_1.$

The locus of $P:(x_1, y_1, z_1)$ is therefore

$$a(b^2-c_2)\,x=b\,(c^2-a^2)y=c\,(a^2-b^2)\,z.$$

7. The normal at a point P on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ meets the coordinate planes is G_1, G_2, G_3. Find the locus of P if

$$PG_1^2+PG_2^2+PG_3^2+=k^2. \qquad \text{(U. P. C. S., 1971)}$$

Ans. The intersection of the surface $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$,

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=\frac{k^2}{a^4+b^4+c^4}.$$

Do from book.

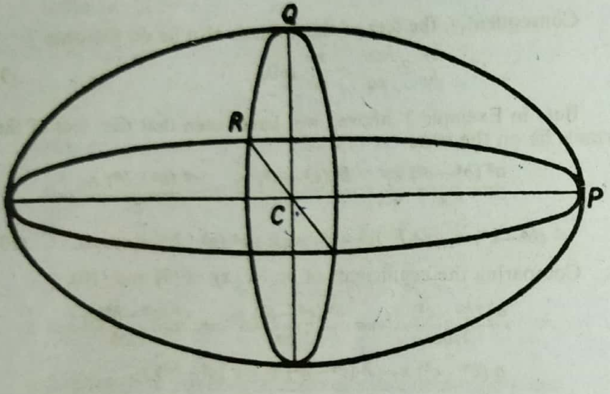
18.9 Conjugate diameters. Let P be a point (x_1, y_1, z_1) on the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ whose centre is C. The equation of the diametral plane of CP is (see § 18.4)

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0.$$

If $P:(x_2, y_2, z_2)$ be a point of the ellipsoid which lies in the above plane, then

$$\frac{x_2x_1}{a^2}+\frac{y_2y_1}{b^2}+\frac{z_2z_1}{c^2}=0.$$

The above relation shows that if Q is on the diametral plane of CP, then P is on the diametral plane of CQ.



If now the diametral planes of CP and CQ intersect in the diameter CR, R lies in the diametral planes of both CP and CQ and thus both P and Q lie in the diametral plane of CR. The diametral plane of CR is thus the plane CPQ. The planes QCR, RCP and PCQ are the diametral planes of CP, CQ and CR respectively and are called **conjugate diametral planes.** CP, CQ, CR are called **conjugate diameters.** If R be the point (x_3, y_3, z_3), we have following relations :

Conjugate semi diameters.

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1,$$

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1; \qquad \text{...(1)}$$

$$\frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}=1,$$

and

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0,$$

$$\frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}=0, \qquad \text{...(II)}$$

$$\frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}=0.$$

From (I) and (II) we conclude that

$$\frac{x_1}{a},\ \frac{y_1}{b},\ \frac{z_1}{c};\ \frac{x_2}{a},\ \frac{y_2}{b},\ \frac{z_2}{c};\ \frac{x_3}{a},\ \frac{y_3}{b},\ \frac{z_3}{c}$$

can be regarded as the direction cosines of three mutually perpendicular lines. We then have the following additional relations :

$$x_1^2+x_2^2+x_3^2=a^2,$$
$$y_1^2+y_2^2+y_3^2=b^2, \qquad \text{...(III)}$$
$$z_1^2+z_2^2+z_3^2=c^2.$$

$$x_1y_1+x_2y_2+x_3y_3=0,$$
$$y_1z_1+y_2z_2+y_3z_3=0, \qquad \text{...(IV)}$$
$$z_1x_1+z_2x_2+z_3x_3=0.$$

Relation (III) on adding gives

$$CP^2+CQ^2+CR^2=a^2+b^2+c^2.$$

Thus the sum of the squares on any three conjugate diameter is constant.

The coordinate axes are a particular case of conjugate diameters.

18.91 Solved Examples.

1. *CP, CQ, CR are three conjugate semi-diameters of an ellipsoid. Show that the sum of the squares of the areas QCR, RCP, PCQ is constant.*

Let l_1, m_1, n_1 be the direction cosines of the normal to the plane QRC, and let A_1 be the area QCR. Projecting A_1 on the plane $x=0$, we obtain

$$l_1A_1=\frac{y_2z_3-z_2y_3}{2}. \qquad \text{...(I)}$$

It is assumed here that the coordinates of P, Q, R, are (x_r, y_r, z_r), $r=1, 2, 3$.

From relation (IV) of Art. 18·9, we have

$$\frac{\frac{x_1}{a}}{\frac{z_2y_3-y_2z_3}{bc}}=\frac{\frac{x_2}{a}}{\frac{z_3y_1-y_3z_1}{bc}}=\frac{\frac{x_3}{a}}{\frac{z_1y_2-z_2y_1}{bc}}$$

$$=\pm\sqrt{\frac{\Sigma \frac{x_1^2}{a^2}}{\Sigma\left(\frac{z_2}{c}\cdot\frac{y_3}{b}-\frac{y_2}{b}\cdot\frac{z_3}{c}\right)^2}}=\pm 1.$$

Thus

$$\frac{x_1}{a}=\pm\frac{z_2y_3-y_2z_3}{bc}$$

Hence, from (I),

$$l_1A_1=\pm\frac{bcx_1}{2a}.$$

Similarly, $m_1A_1=\pm\frac{cay_1}{2b}$, $n_1A_1=\pm\frac{abz_1}{2c}$. Further if l_2, m_2, n_2; l_3, m_3, n_3 be the direction cosines of normals to RCP and PCQ, and A_2, A_3 the corresponding areas, we similarly obtain

$$l_2A_2=\pm\frac{bcx_2}{2a},\quad m_2A_2=\pm\frac{cay_2}{2b},\quad n_2A_2=\pm\frac{abz_2}{2c};$$

$$l_3A_3=\pm\frac{bcx_3}{2a},\quad m_3A_3=\pm\frac{cay_3}{2b},\quad n_3A_3=\pm\frac{abz_3}{2c};$$

Squaring and adding the above relations containing A_1, A_2, A_3, we obtain

$$A_1^2+A_2^2+A_3^2=\frac{1}{4}b^2c^2\Sigma\frac{x_1^2}{a^2}+\frac{1}{4}c^2a^2\Sigma\frac{y_1^2}{b^2}+\frac{1}{4}a^2b^2\Sigma\frac{z_1^2}{c^2}$$
$$=\tfrac{1}{4}(b^2c^2+c^2a^2+a^2b^2).$$

This proves the proposition.

notes:- ~~2~~. *The extremities of the three conjugate semi-diameters of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *are P, Q, R. Find the equation to the plane PQR, and show that it touches another ellipsoid at the centroid of the triangle PQR.*

(*U. P. C. S.*, 1972; *Agra*, 1975; *Rohilkhand*, 1977)

take plane $lx+my+nz=p$.

Let P, Q, R be the points (x_r, y_r, z_r), $r=1, 2, 3$. If the equation to the plane PQR be $ux+vy+wz=d$, we have

$$ux_1+vy_1+wz_1=d, \quad ...(1)$$
$$ux_2+vy_2+wz_2=d, \quad ...(2)$$
and
$$ux_3+vy_3+wz_3=d. \quad ...(3)$$

Multiplying (1) by x_1, (2) by x_2, (3) by x_3, adding and using the relations for three mutually perpendicular lines,

$$ua^2=d(x_1+x_2+x_3).$$

Similarly,

$$vb^2=d(y_1+y_2+y_3)$$
and
$$wc^2=d(z_1+z_2+z_3).$$

Substituting for u, v, w in the equation of the plane we get

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1.$$

This obviously is the equation of the tangent plane to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3},$$

at the point $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$, since

$$\frac{(x_1+x_2+x_3)^2}{9a^2}+\frac{(y_1+y_2+y_3)^2}{9b^2}+\frac{(z_1+z_2+z_3)^2}{9c^2}=\frac{1}{3}.$$

This point is the centroid of the triangle PQR.

Examples on Chapter XVIII

1. Find the equation of the director sphere of a central conicoid. (*Lucknow*, 1979; *I. A. S.*, 1977)

Hint. The director sphere is the locus of the point of intersection of the mutually perpendicular tangent planes. Three mutually perpendicular tangent planes to $ax^2+by^2+cz^2=1$ are

$$l_rx+m_ry+n_rz=\sqrt{\frac{l_r^2}{a}+\frac{m_r^2}{b}+\frac{n_r^2}{c}},\quad r=1, 2, 3.$$

Square and add.

Ans. $x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

~~2~~. Tangent planes are drawn to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

through the point (p, q, r). Prove that the perpendiculars to them from the centre lie on the cone

$$(px+qy+rz)^2=a^2x^2+b^2y^2+c^2z^2.$$

(*I. A. S.*, 1970; *U. P. C. S.*, 1975)

Ex - 2, 3, 5, 9, 11, 13, 15
P. 385 - 387

I.Q
13, 11, 3,

M/A, Ex - 7, 10
P. 386 - 387

3. Find the locus of the mid-points of chords of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ which are parallel to $x=0$ and touch the sphere $x^2+y^2+z^2=k^2$.

Ans. $c^4y^2(x^2+y^2-k^2)+b^4z^2(x^2+z^2-k^2)=2b^2c^2y^2z^2$.

4. OP, OQ, OR are three radii of an ellipsoid whose centre is O and which are mutually at right angles to one another. Show that the plane PQR touches a sphere. (*Lucknow Hons.*, 1962)

5. The section of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane $z=0$ is a parabola. Find the locus of its vertex. (*Lucknow*, 1978)

vertex of cone

Ans. $z=\pm c$.

6. Prove that if the normals at the extremities of a chord of an ellipsoid intersect, the chord is at right angles to its polar with respect to the ellipsoid.

Hint. Two straight lines intersect if they are coplanar.

The polar is the line of intersection of tangent planes at the extremities of the chord.

7. Obtain the tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which are parallel to the plane $lx+my+nz=0$.

If $2r$ is the distance between two parallel tangent planes to the ellipsoid, prove that the line through the origin and perpendicular to the planes lies on the cone

$$x^2(a^2-r^2)+y^2(b^2-r^2)+z^2(c^2-r^2)=0. \qquad (\textit{Kanpur}, 1977)$$

8. A pair of perpendicular tangent planes to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

passes through the fixed point $(0, 0, k)$. Show that their line of intersection lies on the cone

$$x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$

(*Delhi Hons.*, 1959)

9. If the plane $lx+my+nz=p$ passes through the extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Prove that $a^2l^2+b^2m^2+c^2n^2=3p^2$. (*Lucknow*, 1978)

10. Show that the locus of the foot of the perpendicular from the centre to the plane through the extremities of the conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is

$$a^2x^2+b^2y^2+c^2y^2=3(x^2+y^2+z^2)^2.$$

(*Sagar*, 1966; *Ravishankar*, 1970)

11. Prove that the pole of the plane through the extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lies on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=3$. (*Jabalpur* 1968)

12. OP, OQ, OR are conjugate diameters of an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. At Q and R tangent lines are drawn parallel to OP and p_1, p_2 are their distances from O. The perpendicular from O to the tangent plane at right angles to OP is p. Prove that

N. Imp

$$p^2+p_1^2+p_2^2=a^2+b^2+c^2.$$

13. Prove that the locus of the point of intersection of three tangent planes to $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$, which are parallel to conjugate diametral planes of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}.$$

$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=\sqrt{\frac{\alpha^2x_1^2}{a^4}+\frac{\beta^2y_1^2}{b^4}+\frac{\gamma^2z_1^2}{c^4}}$

14. Show that anyone of three equal conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

lies on the cone whose equation is

$$(a^2+b^2+c^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=3(x^2+y^2+z^2).$$

15. Find the locus of the vertices of the enveloping cones of the surface $x^2/a^2+y^2/b^2+z^2/c^2=1$, if the sections of cones by the plane $z=0$ are circles. (*U. P. C. S.*, 1973)

Ans. $x=0$, $\frac{y^2}{b^2-a^2}+\frac{z^2}{c^2}=1$;

$y=0$, $\frac{x^2}{a^2-b^2}+\frac{z^2}{c^2}=1$.

16. If P, Q, R and P', Q', R' are the feet of the six normals from a point to the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and the equation of the plane PQR be $lx+my+nz=p$, show that the equation of the plane $P'Q'R'$ is

$$\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0.$$

(*Punjab*, 1971; *I. A. S.*, 1976)

387

APPENDIX

VOLUME OF A TETRAHEDRON

To find the volume of a tetrahedron in terms of the coordinates of its vertices.

Let (x_r, y_r, z_r), $r=1, 2, 3, 4$ be the coordinates of the vertices A, B, C, D of a tetrahedron. The equation to the plane ABC is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0,$$

i.e.,
$$x\Delta+y_1\Delta_2+z\Delta_3=\Delta_4 \qquad ...(1)$$

where

$$\Delta_1\equiv\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \text{ etc.}$$

Equation (1) can also be written as

$$x \cos \alpha+y \cos \beta+z \cos \gamma=p. \qquad ...(2)$$

If $\triangle$ be the area ABC, its projections on the planes $x=0$. $y=0$, $z=0$ are $\cos \alpha . \triangle$, $\cos \beta . \triangle$ and $\cos \gamma . \triangle$.

The projection on the plane $x=0$ is also equal to

$$\frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}.$$

Hence

$$\cos \alpha . \triangle=\frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 0 \end{vmatrix}$$

with similar expressions for $\cos \beta . \triangle$ and $\cos \gamma . \triangle$.

We can now write equation (1) as

$$2\Delta\,(x\cos\alpha+y\cos\beta+z\cos\gamma)=\Delta_4=2p\Delta.$$

Now $\frac{1}{3}p\Delta$ is the absolute measure of the volume of the tetrahedron $OABC$.

We thus get volume $OABC$

$$=\frac{1}{6}\Delta_4=\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

To get the volume of the tetrahedron ABC, we transfer the origin to D. The volume of the terahedron is then the absolute value of the determinant

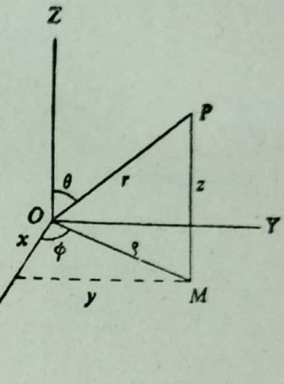
$$\frac{1}{6}\begin{vmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 \end{vmatrix}$$

i.e.,
$$\frac{1}{6}\begin{vmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 & 0 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 & 0 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 & 0 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

i.e.,
$$\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

CYLINDRICAL AND SPHERICAL COORDINATE SYSTEM

In space, we often make use of two other coordinate systems known as cylindrical and spherical systems. These two systems may be regarded as generalizations in space of the polar coordinate **system in plane.**



Let P be a point in space whose coordinates referred to rectangular axes OX, OY, OZ are (x, y, z). Let M be the projection of P on the xy-plane. If the polar coordinates of M in the xy-plane be (ρ, ϕ), we say that the **cylindrical coordinates** of P are (ρ, ϕ, z).

The surface ρ=constant is a right circular cylinder having the z-axis as its **axis,** ϕ=constant is a plane passing through z-axis and z=constant another plane parallel to xy-plane. The rectangular coordinates $(x, y\ z)$ and the cylindrical coordinates (ρ, ϕ, z) of the same point P are related by the equations

$$x=\rho\cos\phi,\ y=\rho\sin\phi,\ z=z.$$

If the distance OP be equal to r, and the angle ZOP be equal to θ, we say that the **spherical polar coordinates,** or simply **polar coordinates,** of P are (r, θ, ϕ). The surface r=constant is a sphere with centre at the origin (or pole) O, the surface θ=constant is a right circular cone having the z-axis as its axis and the origin as its vertex and ϕ=constant is a plane passing through the z-axis. The relation between rectangular coordinates (x, y, z) and polar coordinates (r, θ, ϕ) of the point P is easily obtained as

$$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta.$$